Information Flow in Synchronization

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1. Introduction

Chaos synchronization is a well-known mechanism that creates structure in complex dynamical systems [Pecora 1990]. While much can be said of specific cases, little is known about the fundamental limits and robustness of synchronization. Here we review recent efforts to establish a fundamental theory of chaos synchronization on information-theoretic grounds. Chaotic oscillations are characterized by positive entropy [Shaw 1980]; thus, synchronized oscillators must share a common information source. Where is this information generated and how does it flow between one dynamical element and the next? For unidirectional coupling, information generated by the drive oscillator must be encoded and transmitted through a coupling channel to a response oscillator, where it is decoded and used to entrain the dynamics. The process is essentially a communication problem, and it follows that Shannon's communication theory can be applied to the phenomenon of chaos synchronization [Shannon 1948]. The connection can be made explicit with the use of *symbolic dynamics*, which transforms a chaotic flow into a discrete sequence of symbols with transition rules [Kitchens 1998]. Recently, symbolic dynamics was used to show that a channel capacity exceeding the Kolmogorov-Sinai entropy of the drive system is theoretically necessary and sufficient to sustain synchronization to any precision [Stojanovski 1997]. In this paper we build upon this result and derive some fundamental limits of synchronization quality with regard to detector accuracy and time delay. We demonstrate these new results in an experimental system using chaotic electronic circuits designed to explore synchronization near the information limit [Pethel 2003]. We conclude by sketching a roadmap toward a more general theory of mutually coupled systems. On the whole, a collection of coupled chaotic oscillators can be thought of as a single complex spatio-temporal system with the phenomenon of synchronization causing the appearance of structure. We posit, then, that a general theory of synchronization based on symbolic dynamics can serve as a departure point for a theory of complexity.

2. Synchronization

We consider the synchronization of two identical chaotic oscillators, labeled drive (D) and response (R), connected by a unidirectional coupling channel. For synchronization, information generated by the chaotic dynamics of D must be communicated to R by the channel. According to communication theory, the elements of a communication system include an information source, a transmitter that encodes the information for transmission through a channel, a receiver that receives and decodes the transmission, and a destination that makes use of the information [Shannon 1948]. In Fig. 1 we cast unidirectional synchronization similarly. The information source D is coupled to a channel by means of a detector, and the channel output is connected to the destination R by a controller. Practically, we say the oscillators are synchronized when the dynamics of R match those of D to a specified fidelity. Typically, one compares D and R at the same time; however, one may also consider so-called *achronal synchronization*, in which R lags or leads D by a fixed amount of time [White 2002].

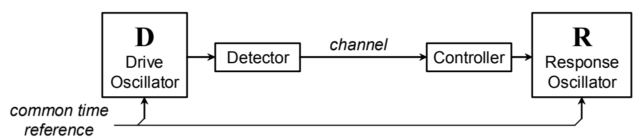

Figure 1. Drive-response configuration for investigating synchronization with unidirectional coupling.

To provide an intuitive understanding of the information transmission required to sustain synchronization, we suppose D and R are synchronized at t_i with quality N_b, where N_b is the number of bits of precision. Given this initial condition an uncontrolled R will match the next return at t_{i+1} with a precision slightly less than N_b bits: the loss in precision is due to the divergent flow inherent to chaotic dynamics. Therefore, to maintain synchronization, the communication system theoretically need transmit only the bits required to make the predicted value of D at t_{i+1} fully N_b bits accurate again. The rate at which this information must be sent is simply the Kolmogorov-Sinai entropy of the system, and the theory of symbolic dynamics provides a transformation that allows us to isolate the information corresponding to these last bits of precision [Stojanovski 1997].

3. Symbolic Dynamics

Symbolic dynamics is used to represent the continuous trajectory of a chaotic oscillator using discrete samples with finite precision [Collet 1980]. To generate a symbolic representation of a trajectory, the state space containing the chaotic attractor is partitioned into regions, each of which is labeled with a symbol. Whenever the state enters a new region the corresponding symbol is generated; for flows represented by a return map, a new symbol is generated for each return using a partition of the return surface. In this way any trajectory is mapped to a bi-infinite symbol sequence $\mathbf{s} = \ldots s_{-2}\, s_{-1}\, .\, s_0\, s_1\, s_2 \ldots$, where s_0 is the current symbol, future symbols are written toward the right, and past symbols to the left. By construction, every trajectory on a chaotic attractor produces a unique symbol sequence. Likewise, every initial condition x corresponds to a trajectory, which can then be mapped to sequence space. We denote this mapping by the coding function, $\mathbf{s} = r(x)$ [Hayes 93].

State space and sequence space are topologically equivalent provided the mapping r is one-to-one and continuous. That is, every observed system state corresponds to a unique symbol sequence and nearby points in state space are also nearby in sequence space. For two sequences $\mathbf{s} = \ldots s_{-2}\, s_{-1}\, .\, s_0\, s_1\, s_2 \ldots$ and $\boldsymbol{\sigma} = \ldots \sigma_{-2}\, \sigma_{-1}\, .\, \sigma_0\, \sigma_1\, \sigma_2 \ldots$ we define the distance between them by [Devaney 1989]

$$d(\mathbf{s},\boldsymbol{\sigma}) = \frac{1}{2}\sum_{i=-\infty}^{\infty}\frac{\delta(s_i,\sigma_i)}{2^{-|i|}} \tag{1}$$

where

$$\delta(s_i,\sigma_i) = \begin{cases} 0, & s_i = \sigma_i \\ 1, & s_i \neq \sigma_i \end{cases} \tag{2}$$

Thus sequences $\mathbf{s}$ and $\boldsymbol{\sigma}$ are nearby if they agree in the first few symbols about the current symbol. A partition that guarantees these properties is *generating*, and finding a generating partition is straightforward for one-dimensional maps but much more difficult in two or more dimensions [Hao 1998, Davidchack 2000, Kennel 2003]. For the following we merely assume that such a partition exists.

A key benefit of the symbolic description is that it converts a complex dynamical system into a trivial shift map. That is, if the ith return is $\mathbf{s}_i = r(x(t_i))$, the next return $\mathbf{s}_{i+1} = r(x(t_{i+1}))$ is computed by simply shifting each symbol one position to the left (or, equivalently, moving the period one symbol right). In practice, we observe only a truncated representation of $\mathbf{s}_i$ consistent with our detector resolution, and the shifting process introduces new information that appears on the right hand side of the truncated sequence. To illustrate, we consider the finite symbolic description $\mathbf{s}_i$ = "10.1101110". The next return is $\mathbf{s}_{i+1}$ = "01.101110?", where "?" is a new symbol that cannot be predicted based on the limited knowledge of $\mathbf{s}_i$. One can think of this symbol as a previously undetectable detail of the initial condition that is now revealed by the chaotic dynamics. The shifting process is illustrated in Fig. 2, where a hypothetical sequence of finite symbolic returns are shown. In this process, a new symbol of information appears on the right with each return.

With regard to synchronization, the symbolic representation immediately shows that the channel need only transmit one symbol per return to maintain synchronization between D and R to any level of precision. Only one new symbol is generated per return, which must be detected at D and used to control R, as shown in Fig. 3. On average, this new symbol contains H bits of information, where H is the Shannon

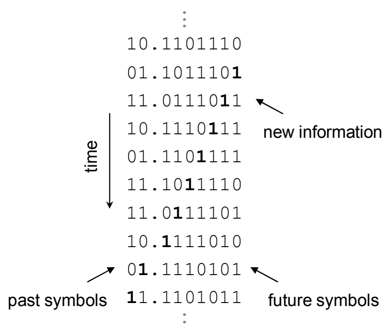

Figure 2. Symbolic dynamics of the shift map applied to a truncated symbolic representation of the system state.

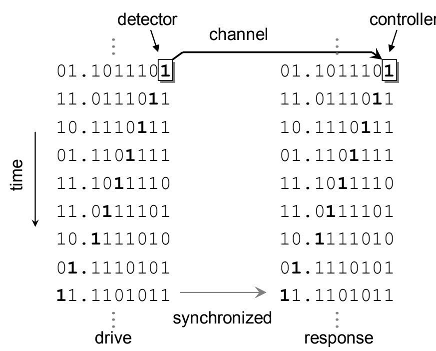

Figure 3. Communication process showing only one symbol per return communication required to sustain synchronization.

entropy of the system. In this context, H is related to the Kolmogorov-Sinai entropy in bits per second by including the average symbol rate [Shannon 1948]. Synchronization cannot be maintained if less than this amount of information is transmitted per cycle, and in the rest of this paper we discuss factors that influence synchronization quality once this minimum condition is met.

4. Synchronization Quality

To assess synchronization quality, we let $\mathbf{s}$ and $\boldsymbol{\sigma}$ denote the current symbolic state of D and R, respectively. We say $\mathbf{s}$ and $\boldsymbol{\sigma}$ agree to m symbols when $d(\mathbf{s}, \boldsymbol{\sigma}) \leq 2^{-m}$. As such, m quantifies the synchronization quality, and topological equivalence guarantees that D and R rapidly approach each other in both sequence space and state space as m is increased. On average, R contains $Q_S = mH$ bits of information about D. Synchronization of this quality is maintained when the controller produces a

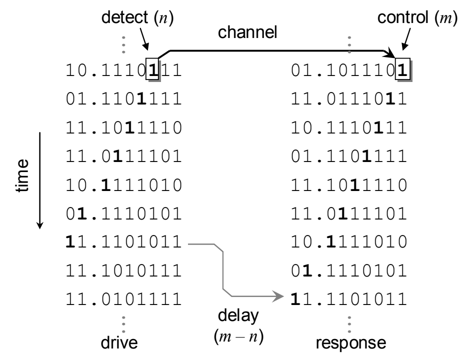

Figure 4. "Achronal" synchronization achieved by detecting and controlling information at different relative precision.

perturbation such that the m^{th} future symbol of D appears in the m^{th} position of R. The cost of increasing m appears in the detector resolution: to detect the m^{th} symbol the detector must be able to extract $Q_D = mH$ bits of information about the drive state. For true synchronization R cannot contain more information than the detector is capable of extracting from D; thus, $Q_S \leq Q_D$.

In general, we can consider the case where we detect and transmit the n^{th} future symbol of D, while we control at the m^{th} future symbol in R. For $n \neq m$, R will be controlled to the same symbols as D, offset by a fixed number of returns. The resulting D and R waveforms will be time shifted by $L = m - n$ cycles, as shown in Fig. 4. For $m > n$ the response lags behind the drive—a situation referred to as *lag synchronization*. In this case the quality of synchronization exceeds the detector resolution. The opposite scenario can be created as well. For $n > m$ the lag is negative and the response leads the drive. This effect has been seen previously and has been described as *anticipating synchronization* [Voss 2000]. In sequence space we can clearly see that an upper bound on synchronization quality is $m \leq n + L$. Multiplying by H, we get the equivalent statement in an information sense:

$$Q_S \leq Q_D + LH \tag{6}$$

This bound on synchronization quality, which is valid for identical oscillators connected by a unidirectional and noiseless channel, illuminates a fundamental tradeoff between detector precision and achronal synchronization. Namely, synchronization quality can be enhanced by lag, even to the point that it can exceed the detector precision.

Experiments

We now present recent experimental results that test the information limits of synchronization [Pethel 2003]. We use two nearly-identical electronic circuits that exhibit chaotic, simply folded band dynamics, and successive local waveform maxima are well approximated by a unimodal, one-dimensional return map. As such, the circuit dynamics are described by a symbolic dynamics of two symbols, using a generating partition corresponding to the map maximum. Using the standard labeling, every waveform peak below the threshold generates a “0”, and every peak above generates a “1”. The oscillators are coupled via a channel that transmits only symbols from D to R. The detector uses a high-precision analog-to-digital converter and a look-up table to predict the m^{th} future symbol of D. Concurrently, dynamic limiting is used to control the n^{th} future symbol of R [Corron 2002a, 2002b, 2003].

In this system, synchronization is achieved by transmitting only one bit of information per return. Since the Kolmogorov-Sinai entropy of this oscillator is ~0.7 bits/return, the channel capacity is theoretically sufficient to maintain synchronization. The experimental apparatus allows us to independently adjust the detector precision m and controller precision n to explore the synchronization near

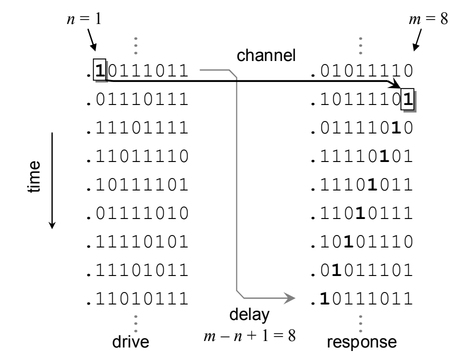

Figure 5. Lag or delay synchronization achieved by detecting D with $n = 1$ symbol precision, and R is controlled at the $m = 8$ future symbol.

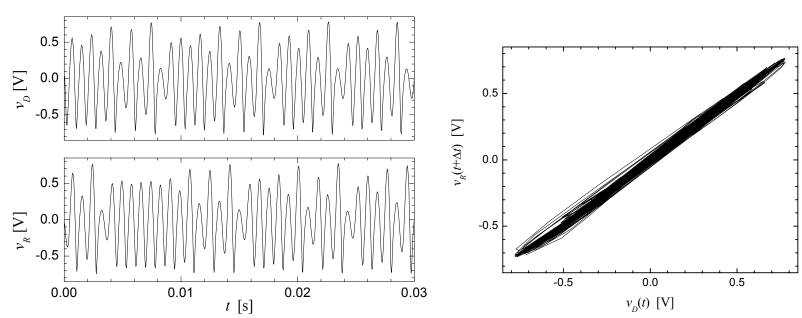

Figure 6. Drive (v_D) and response (v_R) waveforms measured from electronic circuits showing synchronization with an 8-cycle lag ($\Delta t = 6.7$ ms). D is detected with $n = 1$ bit of precision, and R is controlled at the $m = 8$ future bit.

this theoretical information limit.

To demonstrate lag synchronization, we detect the current symbol ($n = 1$) and control at the 8th future symbol ($m = 8$) as shown in Fig. 5. The experimental results in Fig. 6 shows R synchronizes to D with a significant delay. Details of the experiment require an extra cycle of processing to apply control; thus, the total delay is $L + 1 = 8$ returns. According to equation (6), high-quality synchronization is achieved despite the poor detector quality by exploiting a significant lag.

For anticipating synchronization, we detect D with high precision ($n = 9$) and control R at only the 6th future symbol ($m = 6$) as shown in Fig. 7. Since $n > m$, we now expect a negative delay ($L + 1 = –2$). In Fig. 8, the experimental results show that R leads D by 2 cycles. This state of anticipation is achieved by exploiting the high-precision detector at the cost of synchronization quality.

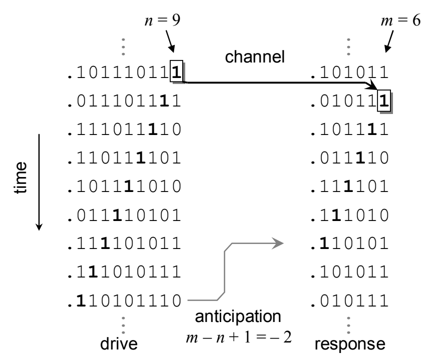

Figure 7. Lead or anticipating synchronization achieved by detecting D with $n = 9$ and controlling R at $m = 6$.

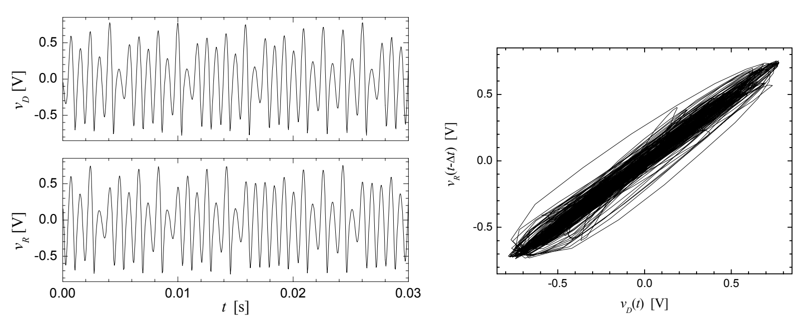

Figure 8. Drive (v_D) and response (v_R) waveforms for synchronization with a 2-cycle anticipation ($\Delta t = 1.7$ ms). D is detected with $n = 9$, and R is controlled at $m = 6$.

6. Conclusion

The symbolic dynamical model of synchronization illuminates fundamental relationships with regard to channel capacity, detector accuracy, synchronization quality, and lag. The model is easily extended to consider synchronization among disparate systems [Corron 2002b]. The circuit results show that this understanding is robust under experimental conditions and provides an excellent description of simple chaotic flows. The next step in this development is to include mutually coupled systems. The simplest approach is to consider two diffusively coupled unimodal maps. In this case, a four-symbol partition immediately suggests itself—it is merely the product of the solitary map partitions. This choice is trivially correct for the case of perfect synchronization; we will discuss its general properties in a future publication. It is hoped that this scheme will be adequate even for large numbers of coupled maps. Finally, given a satisfactory partition, rules for the admissibility of symbol sequences need to be formulated. These rules can be defined empirically, but a functional relationship with a global parameter—such as coupling strength—must be found. This is a critical step because it allows us to catalog all the available dynamical structures as a function of this global parameter, which is the most we can reasonably ask from a theory of complexity.

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