THE DYNAMIC OF PRODUCT-POTENTIAL SOCIAL SYSTEMS AND REPRESENTATION THEORY

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Abstract

The product potential system is a locally interacting system with a potential (defined by requiring that a product-integral along any closed path on a graph equals to an identical transformation) system of psychological reactions (consisting of marks or fields on the edges of the graph of relations). A locally interacting process for the product potential system of relations can be given by an algebraic representation of an process of multiplication on the randomly chosen so call "control matrix". We found one to one maps between thermodynamic states of system (the thermodynamic state for the system is measure) and so call "left ideals" on the semi-group of control matrices.

Key words: balanced groups, locally interrupting systems,nets of automaton, spatial Markov's chains, product-potential system, representation.

The algebraic description of the dynamics of the states for potential systems. The dynamics of the states of a net of automata was determined previously [1] by a family of conditional probabilities. But in the case when group of relations is a **complete graph** we will find the algebraic description of the dynamics of the states. In the models, which will be presented in this paper, the dynamics of states are controlled by a subset of a stochastic matrices so call set of "control matrices". The set of stochastic matrices M_s are matrices with the integer entries equal to 0 or 1 and with sum of all rows equal one.

Let M is set of control matrices, where control matrices are stochastic matrices with zero elements on the main diagonal (Tr(A)=0). Elements of M we will call *the control matrices* .

Let M^* (M_s^*) is set of finite length words in alphabet M (M_s). So M^* is set of words A_m $A_{(m-1)}$ $\ldots$ A_1, where A_1, A_2, $\ldots$, A_m is matrices from M (M_s). So "word" is productions of m matrices for any not negative m. The sets of words M^* and $M*_s$ are algebraic semigroups.

Matrix of reaction Rg presents the product potential system reactions on the complete graph: g(k,l) is reactions on the edge (k,l) and g(s,s)=e, where e is identical transformation (ex=x for all x). Potentiality on the complete graph means that g(k,l)g(l,s) g(k,s) = e, g(k,l)g(l,s)=e for any three nodes k, l, s. Therefore, g(k,l)g(l,s) g=g(k,s) because of property of potentiality and gg=e.

We will show that local interacting process for product potential system of reactions can be defined as algebraic presentation set of control matrices into semigroup of transformations.

The representation is map F of algebraic semi-group H into the group of liner matrices GL, where for any two arbitrary elements from H property F(h_1h_2) = F(h_1)F(h_2) hold (F(h_1) and F(h_2) are linear matrices).

Right now we show connection between old description of the dynamic and new one on the particular example.

Suppose all edges of graph of relations are marked by the elements belonging to the group of reactions G in accordance with the given system of reactions and the choice functions. For the arbitrary finite group of relations the system of elements marking a graph of relations can be written in the form of a square matrix. The dynamics of the system after n steps is determined by a product of n matrices applied to the initial state of the net, where every matrix is a Kronecker's product of the control matrix and the square matrix of the transformations (every entry in this matrix is a transformation of the state space). If we are given a system of the conditional probabilities on the Kronecker's product matrix, then the dynamics of the network's state will be defined.

Let A = ($a_{i,j}$, where $1 \leq i$, $j \leq n$) and B = ($b_{i,j}$, where $1 \leq i$, $j \leq n$). The matrix A*B = ($a_{i,j}$B, where $1 \leq i$, $j \leq n$) is called a Kronecker's product of the matrices A and B.

We call semigroups A(M^*)=(M^*)*Rg={ w*Rg for all finite words from M^* } and A(M_s^*)=(M_s^*)*Rg={ w*Rg for all finite words from M_s^* } semigroups of transformations(operators). The elements of matrices of transformations are reactions. For instances it will be 2x2 matrices (see [1]).

The representation of the subgroup and description of the states dynamics.

The homomorphism $\rho : M^* \to (M^*)*Rg$ is a representation of the semigroup M^* into the semigroup of operators A(M^*), where $\rho(w)$=w*Rg, for any word w from) M^*.

Example. Let $C_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $C_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *and* $C_2C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Therefore $(C_2 * Rg)(C_1 * Rg) = \begin{pmatrix} 0 & g_{1,2} & 0 \\ 0 & 0 & g_{2,3} \\ 0 & g_{3,2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & g_{1,3} \\ g_{2,1} & 0 & 0 \\ 0 & g_{3,2} & 0 \end{pmatrix} = \begin{pmatrix} g_{1,2}g_{2,1} & 0 & 0 \\ 0 & g_{2,3}g_{3,2} & 0 \\ g_{3,2}g_{2,1} & 0 & 0 \end{pmatrix}$ $= \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ g_{3,1} & 0 & 0 \end{pmatrix}$.

We use potentiality of field $g_{i,j}$: $g_{1,2}g_{2,1}$ =e, $g_{2,3}g_{3,2}$ =e, and $g_{3,2}g_{2,1} = g_{3,1}$.

Contrariwise $(C_2C_1) * Rg = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} * \begin{pmatrix} e & g_{1,2} & g_{1,3} \\ g_{2,1} & e & g_{2,3} \\ g_{3,1} & g_{3,2} & e \end{pmatrix} =$

$\begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ g_{3,1} & 0 & 0 \end{pmatrix}$.

So $\rho(C_2C_1)$ =$(C_2C_1) * Rg$=(C_2*Rg) (C_1*Rg) =$\rho(C_2)$ $\rho(C_1)$.

In general case very easy to prove that homomorphism $\rho : M^* \to A(M^*)$ ($\rho : M_s^* \to A(M_s^*)$) is a representation of the ring M^* (M_s^*) into the ring of operators (A(M^*) (A(M_s^*)).

We call set I **left ideal** if for any word **w** **w**I belongs to I (**w**I ⊂ I).

The next theorem will be base for Theorem 1.

Theorem 1. The semigroup M^*(N) (M_s^*(N)) has N ideals I(1), ... , I(N), where matrix I(k) has only non zero column: all elements of column k equal one (1 ≤ k ≤ N).

Proof of the theorem 1.

Suppose, that matrices I(1), ... , I(N) belong to M^*(N) (M_s^*(N)). We will prove it below. Note that all words from $M^*(N)$ or $M_s^*(N)$ are matrices with sum of all rows equal one, where N is size of square matrices. It means that for any word **w** **w**I(k)= I(k) (1 ≤ k ≤ N). Particulary, I(j)I(k)=I(k).

Representation of the semigroup and dynamics of a state of a potential system. Graph of transitions for the Markov chain.

Now we are ready to describe of the dynamics of a state of a potential system.

For describing the dynamics of a state of potential system the rings M^* and A(M^*) and almost representations $\rho : M^* \to A(M^*)$ will be used. The dynamics of the states can be realized as representation of a random product process on the semigroup M^* into the semigroup A(M^*). It means that first of all we get randomly chosen initial ward (matrix) C_0 from M^*. Then we multiply initial word (matrix) on the randomly chosen matrix from M and so on. All steps are independent. Throughout n steps we get random trajectory C_0 , C_1C_0 , $C_2C_1C_0$, ... , $C_nC_{n-1} \ldots C_1C_0$ as a result of the random product process on the semigroup M^* . Then we map our trajectory on the semigroup M^* into the ring A(M^*) and get real trajectory

$$\rho(C_0), \rho(C_1C_0), \ldots, \rho(C_nC_{n-1} \ldots C_1C_0)$$

, where $\rho(C_nC_{n-1} \ldots C_1C_0) =$

$(C_n * Rg)(C_{n-1} * Rg) \ldots (C_0 * Rg) = \rho(C_n)\rho(C_{n-1}) \ldots \rho(C_0)$.

For an arbitrary initial word (matrix) with probability one we have to reach one of the ideals of the ring M^*. Therefore, according to theorem 1 we must examine the behavior of our system in the ideals. It hardly simplifies our problem.

Theorem 2. The number of invariant measures of the random product process equals to the number of ideals of the ring A(M^*) (A(M_s^*)).

Proof of the theorem 2. The theorem 1 means that stochastic product process on M^* (M_s^*) with probability one converges to one of N ideals. We map our trajectory that converge to ideal into ring A(M^*). The map ρ is representation and image of ideals I(1), ... , I(k) are ideals in ρ(A(M^*)): $\rho(\mathbf{w})\rho$(I(k)) = $\rho(\mathbf{w}$I(k))= ρ(I(k)). It means that ρ(I(1), ... , ρ(I(N) are left ideals of ρ(A(M^*)). The ideals $\rho(I(k))$ is matrix with zero columns 1, ... , k-1, k+1, ... ,N and nonzero column k is $(g_{k,1}, \dots ,g_{k,N})^t$. The ideal $\rho(I(k))$ can be reached for some number of steps. The ideals $\rho(I(k))$ represent final classes and system has to reach one of N final classes and never leave them.

Main structural lemma. An arbitrary element $D \in A(M,N)$ can be represented as

$$D = P_1 S^{n_1} P_2 S^{n_2} P_3 \dots P_{n-1} S^{n_{n-1}} P_n$$

, where S = $\{c_{i,j} = 1$, for i = j = 1 , i = k , j = k + 1 , where k = 2 , . . . , n and all other elements are zero $\}$ and $detP_j \neq 0$, for all j = 1,2, . . . ,n (it means that every matrix P_j represents a permutation).

We will use the language of orbits for analysis of structure of the semigroup M_s^* to prove the lemma.

The language of the orbits. Suppose that M_s (N) is set NxN stochastic (control) matrices.

Let a set of matrices which have one column with m_1 nonzero elements, one column with m_2 non zero elements, and so on and one column with m_n nonzero elements , where $m_1 \geq m_2 \geq \dots \geq m_n \geq 0$ be denoted by $[m_1, m_2, \dots, m_n]$ and call it **class**.

We represent the semigroup M_s^* as union of classes $[m_1,m_2,\dots , m_N]$.

The level of the class $[m_1, m_2 , \dots , m_k, 0 , \dots , 0]$ is k ($m_1 \geq m_2 \geq ... \geq m_k > 0$, and $m_1 + m_2 + \dots + m_k$ =N).

Note. The only class of ideals [N, 0, . . . , 0] has level one.

Let us denote by symbol PG(N) the group of the permutations of the N standard vectors e(1), . . . , e(N), where e(k) = $(0, \dots ,0,1, 0, \dots ,0)^t$, k=1, . . . ,N. The permutation P(j,k) transforms e(j) and e(k). We then see that class [1, . . . ,1] and group of permutations PG(N) are same.

We can define the group PG(N)× PG(N) actions on arbitrary elements B of A(B) as next actions: $(P_1, P_2)B = P_1 B P_2$.

The **orbit** of arbitrary elements B is set O(B) = $\{$ $P_1 B_2$ for all P_1 and P_2 from group PG(N)$\}$.

By ordinary way we can define the subgroup Stabl(B) =(P_1,P_2): $P_1 B P_2$ = B. Stabl(B) is a subgroup of group PG(N)× PG(N).

All our classes are the orbits. For instance class [2, 1, 0] = O(C), where $S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Similarly, the class $[m_1, m_2, \dots , m_k, 0, \dots , 0]$ ($m_1 \geq m_2 \geq \dots \geq m_k > 0$ and $m_1 + m_2 + \dots + m_k$ = N) is the orbit of one element. We call this elements "canonical". Normally we will use canonical element S that has k nonzero columns and N - k zero columns. The first column has the first m_1 units and the rest of elements equal to zero. The second column has first m_1 elements equal to zero, then m_2 elements equal 1, and the rest of elements equal zero again and so on.

Very easy to calculates the number of elements in orbits: $|O(B)| = |G(N)|^2/|Stabl(B)|$, where $|G(N)|$ = N!.

An any class is orbit and set of words M_s^* is the union of orbits.

Examples. Suppose N=3. In this case we have three classes: [1,1,1], [2, 1, 0], [3, 0, 0], where [1, 1, 1] = G(3), [2, 1, 0] = O(S), where S was described above, and [3, 0, 0] =O(I(1)), where I(1) has only one nonzero column: first column and all elements of it's column equal 1. Let us denote by symbol I(k) elements I(1)P(1,k), where P(1,k) transpose elements one and k (in our case k=2,3). The class orbit [2, 1, 0] or class O(S) has 18 elements. The size of group of permutations of three elements equals 3! =6. The Stabl(S) contains two element: (E,E)and (P(1,2),E), where E is identical permutation, P(1,2) transposes elements one and two. So, $|Stabl(S)| = |(E,E),(P(1,2),E)|$ =2 and $|O(S)|$ = (6·6)/2 =18.

The multiplication of two arbitrary elements from same orbits can belong to same orbits or can be element of same levels' class or lower. This means that early or late the productions fall into lowest level (level 1) with not zero probability.

Example for N=3. If B_1 and B_2 are elements of O(S) it means that exist permutations P_1 , P_2 , P_3, P_4 that B_1 = $P_1 S P_2$ and B_2=P_3SP_4, where S is shift matrix. It is easy to check that S^2 =I(1). Question is when production two elements belong to orbit [3, 0, 0]? We have to find solutions for equation $P_1 \cdot S \cdot P_2 \cdot P_3 \cdot S \cdot P_4$ =I(1) or I(2) or I(3) or $CP_5 S P_4$ belongs to { I(1), I(2), I(3)}.

There are a lot of solutions for this equation: B_1=$P_1 \cdot S \cdot P_3^{-1}$ and B_2 =$P_3 \cdot S \cdot P_4$ for all permutations P_1, P_3, and P_4. The second solutions $B_1 = P_1 \cdot S \cdot P(1,2) \cdot P_3^{-1}$ and $B_2 = P_3 \cdot S \cdot P_4$ are identic to the first one because of $P(1,2)^{-1}$ = P(1,2), P(1,2)S = S.

So it is very likely that productions of two elements of class [2,1,0] will belong to the lower level class [3,0,0].

Right now we prove this lemma only for stochastic matrices: M_s and A(M_s). But this prove is true for NxN control matrices too, where N greater than 4.

Proof of lemma. Suppose B belongs to class [m_1, m_2, ... , m_k, 0, ... ,0], here $m_1 \geq m_2 \geq \ldots m_k > 0$ and $m_1 + m_2 + \ldots + m_k$ = N; It means that B: (x_1, x_2, ... ,x_N) → (x_1, ... ,x_1, x_2, ... , x_2, ... , x_k, ... , x_k, 0, ... ,0).

How easy to see we can find permutation P(1,k) that transform the initial vector (x_1, x_2, ... ,x_N) into (x_k,... , x_1, x_2, ... , x_{k+1}, ... ,x_N). Than we have to apply operator S m_k times and we got (x_k, ... ,x_k, x_1, x_2, ... , x_{k-1}, ... ,x_{N-mk}). Then we must apply P(1,mk+m(k-1) +1) and S m_{k-1} times. We get (x_{k-1}, ... , x_{k-1}, x_k, ... , x_k, x_1, ... , x_{k-2}, ... , $x_{N-m_k-m_{k-1}}$) and so on. Finally we have B =S^{m-1}P(1, m_k+m_{k-1} + ... + m_2+1) ... $S^{m_{k-1}}$ P(1,m_k + k -1) S^{m_k}P(1,k).

Lemma shows us that only multiplication by matrix S leads to ideals; the multiplications by the permutations do not.

The period of time required for our system to reach an ideal is period of convergence and sometime it is very important to evaluate this period. The next theorem (without prove) gives us more information the minimal period of convergence.

Right now we prove that matrices I(1), ... , I(N) and right-shift

matrix S belong to M_s^* and M^*. We prove also that group of permutation are generated by M^*(N), where N grater than three.

Let matrix S is right-shift matrix : S= { s(1,1)=1, s(2,1)=1, s(k,k+1)=1, where k=2, ..., N-2. All another elements equal zero }.

Statement 1. The set of finite words M^* contains shift matrix.

It is mean that shift matrix S can be presented as production of finite number matrices from M. This statement is true for all dimensions.

Let put $A_1 = \begin{pmatrix} O_1 & E_2 \\ E_{N-2} & O_2 \end{pmatrix}$, where E_2 is 2x2 unit matrix, E_{N-2} is (N-2) x(N-2) unit matrix, O_1 and O_2 are matrices with zero elements. Matrix A_1 belong to class [1, 1, ... , 1].

Let $A_2 = \begin{pmatrix} O_3 & E(N-2) & O4 \\ R & O_5 & O_6 \end{pmatrix}$, where E_{N-2} is (N-2)x(N-2) unit matrix, R is 2x1 matrix with all elements equals 1, O_3 and O_4 are (N-2)x1 zero matrices, O_4 is 2x(N-2) zero matrix, and O_6 is 2x1 zero matrix. A_2 belong to class [2,1, ... ,1, 0].

Productions A_1 times A_2 is shift matrix: A_1A_2=S.

Statement 2. The set of permutations from M generates groups of permutations GP(N).

It is mean that M^* contains whole group of permutation. This statement is true for N grater than 3.

Example. For N=3 set control matrix contains only two permutations P(2,3,1) and P(3, 1, 2), where P(2,3,1) P(3, 1, 2) = E, where E is identical permutation ($1 \to 1$, $2 \to 2$, $3 \to 3$). These permutations do not generate group of permutation GP(3).

For N grater than 3 we can prove that transpositions from M (transpositions without stable points) generate all transpositions of two arbitrary elements. The transpositions of two arbitrary elements generate whole transpositions group.

Example. Let n=4. Then product transposition P_1: (i, j, k, s) $\to$ (k, s, j, i) and P_12: (i, j, k, s) $\to$ (k, s, i, j) is transposition P(k,s).

For N=5 product transpositions P_1: (i, j, k, s, t) $\to$ (k, s, i, t, j) and P_1: (i, j, k, s, t) $\to$ (t, k, i, j, s) is transposition P(k,i). Similar constructions work for any even and odd N grater than three.

Conclusion. Statement 1 and 2 (for N grater than 3) show us that prove of main lemma and Theorem 1 and Theorem 2 are true for M^* as well.

For N=3 we prove Theorem 2 by direct calculations.

The set of control matrices M(3) contains two permutations P_1 and P_2 and six non permutations A_1, A_2, A_3, A_4,A_5, A_6, where $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,

$$A_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, A_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

. How easy to check P_1 P_2=E, $A_1 A_4 = A_1 A_3$ =I(1), $A_4 A_1$=A_4 A_2 = I(2), and $A_2 A_6 = A_2 A_5$ =I(3). So E,I(1), I(2), I(3), belongs to M^* and for any matrix B from M^* we can find one C_1 or two matrices C_1 and C_2 that C_1B=I(k) or $C_2 C_1$B=I(k). Last property proves theorem 1.

Note. In this article graph of relations is complete. But we can use this the algebraic method (the method of algebraic description of dynamic of the states for potential system) for arbitrary finite connected graph of relation marked by product potentials system of reactions. For this purpose we have to make our graph complete by adding new edges. Then we will mark new edges by reactions equal product of reactions along any path on the original graph started in source vertex of edges and ending in terminating vertex of edges. The set of control matrices contains only control matrices that less or equal to adjacency matrix of original graph. Similarly the entries of the reaction matrices Rg are reactions on original graph and new ones (built on the added edges). The number of invariant measures will be equal to number of ideals as well. So in general case we have to study the semigroup of words $A(M_s(N,B))$ generated new set of control matrices $M_s(N,B)$ that elements less or equal to elements of adjacency matrix B of original graph. We believe that next theorem is true.

Theorem 3 1. Suppose that the graph of relations is not bipartite. The number of ideals of the ring $A(M_s(N,B))$ equals the number of the vertices and every ideal is generated by one control matrix $I_k = \{t_{k,j} = 1 \ \forall j \in \{1,2,\ldots,n\} \ t_{i,s} = 0 \ \forall \ i \neq k \text{ and } \forall s\} \ k \in \{1,2,\ldots,n\}$.

2.Suppose that the graph of relations is bipartite and $\Gamma = (A,B)$, where $A = A_1 \cup A_2, A_1 \cap A_2 = \emptyset$.The number of ideals of the ring $A_2(B)$ equals $|A_1|\,|A_2|$ and every ideal is generated by two elements S(i,j) and $S^-(i,j)$, where $i \in A_1$, $j \in A_2$ and S(i,j) = ($s_{k,l}$,where $s_{i,l} = 1$, $i \in A_1$, for all $l \in A_1$; $s_{m,j}$ = 1, $j \in A_2$ $\forall m \in A_2$ and ; all other elements are equal 0) , $S^-(i,j)$ = ($s_{i,l}$ = 1, $i \in A_1$, for all $l \in A_2$; $s_{m,j} = 1$, where $j \in A_2$, for all $m \in A_1$ and all other elements are 0).

CONCLUSION

We have proved that a locally interacting process for a product potential system of relations can be represented as an process of multiplication on a randomly chosen control matrix with a transformation into the original process in the end. The main property of a product potential system concerning the number of stable measures (there is a one to one connection between ideals and invariant measures) was derived.

REFERENCES

1. Mayer ,Thomas F. , 1975, *Mathematical Models of Group Structure* . The Bobbs-Merrill Company , Inc., Indianapolis - New York

3. Kovchegov , V.B., 1997, *The potential nets of automata and their application to modelling of human group* . Proceeding of the 15th IMACS World

Congress on Scientific Computation , Modelling and Applied Mathematics , Berlin , August 1997 , Volum 4 (Artificial Intelligence and Computer Science), Wissenschaft and Technik Verlag, Berlin, pp 659-664

4. Kovchegov, V.B., 1993, *A model of dynamics of group structure of human institutions.* Journal of Mathematical Sociology, Vol. 19(1)

5. Kovchegov ,V.B., 1994, *A principle of the maximum nonergodicity for modelling of the human groups by nets of probability automata.* Proceeding of the 14-th IMACS World Congress on Computational and Applied Mathematics , July 11-15 , Georgia Institute of Technology Atlanta , Georgia , USA , pp 787-790

6. Gantmacher F.R., 1974, *The Theory of Matrices* Chelsea, New York