

An application of percolation theory to political science

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Introduction

First we would like to explain **Percolation theory**, and then we refer to the possibility of the application of this theory to political science. There are two kinds of percolation, site percolation and bond percolation. Here we explain them in two-dimensional square lattices (Stauffer & Aharony 1994). We suppose that each site of a very large lattice is occupied randomly with probability P and empty with probability $(1-P)$, and that clusters are groups of neighboring occupied sites. This is site percolation. Then we imagine that each site of the lattice is occupied, and lines are drawn between neighboring lattice sites. Each line can then represent an open bond with probability P , independent of its neighbors, or a closed bond with probability $(1-P)$. A cluster can be formed of these open bonds, and the size of the cluster can be measured. That is bond percolation. Percolation theory concerns itself with these clusters.

At present we cannot identify the exact value of P_c (the percolation threshold) for site percolation on a two dimensional lattice. It is nearly 0.59. But it is known that the exact value of bond percolation is 0.50 in the square lattice. P_c is the probability P of an infinite cluster appearing in an infinite lattice. Here we consider the finite lattice, and if $P > P_c$, we have more than one cluster extending from one side of the lattice to the other, either from above to below or from left to right. We can say that this cluster percolates through the lattice. We use the term “percolate” like: water percolates through a coffee machine. We would like to focus on this phenomenon.

We call these clusters infinite. When the possibility exceeds the threshold, certain qualities of clusters will change drastically. Suppose, for instance, that an ant in the lattice can walk only on the open bonds. When $P < P_c$, that is, all clusters are finite, the ant may be confined within this lattice. But when $P > P_c$, the behavior of the ant will be quite different. It can move from top to bottom, or from right to left in this lattice. In percolation theory, therefore, threshold is the most important factor (Solomon et al. 2000).

Thus percolation theory is used as an opinion dynamics. It explains how an opinion expands and transmits itself in the society. Then, we would like to highlight another aspect, the emergence of the order. In social science, we think of this like that. When one group of supporters of a political belief becomes an infinite cluster within a society,

we can say that it acquires the political power in the society. We can also say that the quality of this group change from minority to majority. That is the exact meaning of “percolate”. The opinion percolates in the society in this sense. We see here the qualitative change of this group.

1. Bethe lattice

Here we have a special case where the exact value of threshold can be solved ; the Bethe lattice or Cayley tree (Stauffer & Aharony 1994). In a small Bethe lattice, each site has z bonds. We deal with the case $z = 3$, so each site has 3 bonds and each bond ends in another site, from which 3 bonds stretch out again. 2 bonds lead to new sites and one bond leads to the origin. Thus these bonds or sites make a cluster, where an origin has three bonds and then, from each site, we can go on in 2 other directions and this stretching process is continued. At the surface of the lattice, the stretching stops. That is the end of the road.

The strong point we should use this theory is that there is no closed loop in this Bethe lattice (see also *ibid.* Chap.2.Sec.4). That means we can go on in new directions and we never go back.

From this point, we can find the percolation threshold in the Bethe lattice easily. We argue here about bond percolation, but the below argument is also valid for site percolation.

We start at the origin site and walk on open bond. When we arrive at the other side of this open bond, we check if there is a chance of finding an infinite path of open bonds. We find $(z - 1)$ new bonds from this site, and if there is more than one bond that is open, we can go on such a bond. Each of these $(z - 1)$ bonds is open with probability p . We have on average $(z - 1)P$ open bonds. If $(z - 1)P < 1$, the probability of finding more than one open bond goes to zero exponentially. Thus we have $P_c = 1 / (z - 1)$ for the value of threshold of the Bethe lattice with z bonds for every site. When $z = 3$, we have $P_c = 0.5$.

2. Model and Theories

We discuss the dual structure of the group within society, positing, for instance, the presence of a relatively small number of intensely active environmentalists in combination with a large yet passive contingent of their supporters. We also consider the effects of limited numbers of terrorists holding strong beliefs in combination with large numbers of relatively passive supporters (Galam 2002).

First, supporters are usually passive, and they do not contact others who are not their neighbors. They make connections between neighbors with some probability and

form groups of communication over certain belief.

Then, activists can influence their neighbors. In the presence of neighbors who are not supporters, an activist can persuade these neighbors to become supporters.

What else can activists do? They can combine groups. Two separate groups both containing activists can merge to form a single larger group through the work of activists.

We have conceived this idea from the concept of **Lotka Volterra equations**. In the previous article (Takahashi 2002), we discussed the relations of two groups with Lotka Volterra (LV) equations. These phenomena can also be discussed in terms of LV equations. LV equations are also used widely in the social science, as the spread of epidemic and the dynamic of combat (ex.Epstein1997). LV equations deal with the populations of groups, but here by approaching the probabilities associated with certain groups, we can obtain equations identical to those of entire populations.

LV equations usually cover antagonistic relations such as those between sharks and fish, but here we can express mutualism through changing signs of some coefficients in the LV equations. Here we should say exactly that this relation is one side mutualism. Only activists can help the number of supporters grow large.

Thus we have arrived at **Bethe Percolation Model** : Here we have a finite Bethe lattice, $z=3$. We think of the sites as citizens. In the first stage, we have some supporters with a belief, but they are so negative. Only with the probability p , neighboring two sites communicate with each other over a belief and then, they connect a communication bond between them. We think of these connections as open bonds. Thus we have closed bonds with the probability of $(1 - p)$, both of whose sites cannot communicate with each other. When an open bond is connected with other open bonds, they form a cluster. Important is the clusters of communication bonds. We would like to know the size of these clusters. Then there are the activists, shown as points (sites) with the probability of q . One of the roles of activists is to recruit supporters. In this model, each point (activist) has less than three bonds (legs), which are persuaded to become supporters' bonds. From this action, we get larger clusters of communication bonds by virtue of the persuasion of activists. Here we should think that an activist is a kind of catalyst, which helps the number of supporters' bonds grow dramatically.

Here we want to generate a graph indicating the probability of activists (q), with respect to the probability of supporters' bonds (p) within a lattice. This graph may show when the biggest cluster of supporters' bond becomes infinite. In each (q', p') , we have carried out simulations 5000 times, and when we find a case in which an infinite cluster emerges more than once, we isolate this value (q, p) . We can then generate a

graph on which to plot these values as points. We have an x-axis (p) graduated in units of 0.005 and a y-axis (q) with units of 0.0025. The operation can also be represented in the following formula:

$$\Pr (q, p) = \Pr (\| C \| = + \infty) > 1/5000$$

$\| C \|$ represents the size of the largest cluster of open bonds (supporters) of the Betha lattice. $\Pr (q, p)$ means the probability of p and q, so this equation represents when an infinite cluster emerges more than once in 5000 trials. Here, in this equation, we use the term “infinite cluster” meaning that the number of the open bonds from the origin to the other end of the connection of bonds is more than 30,000.

We have also captured this idea from that of **Small world network theory**: There are, then, many subgroups among supporters and activists. We refer to these subgroups as clusters. When we want to deal with the relations between clusters, we can use the theory of the small world network. This helps us combine clusters to form larger ones. Many reference materials are available on this subject, e.g. (Watts & Strogatz 1998, Watts 1999).

Small world network theory typically uses a one-parameter model that interpolates between an ordered finite dimensional lattice and a random graph. Small world theory thus modulates between regularity and randomness. It provides a useful demonstration of how a new phase can emerge between the regular and the random. But it cannot account for the nature of relations between activists and supporters, or how the function of networks can change. That is to say, it cannot deal with qualitative changes among clusters.

Thus, we primarily use percolation theory here, with the help of suggestions gleaned from Lotka Volterra equations and small network theory (Moore & Newman 2000, Ahmed & Abdusalam 2000, Newman et al. 2002, Miramontes & Luque 2002).

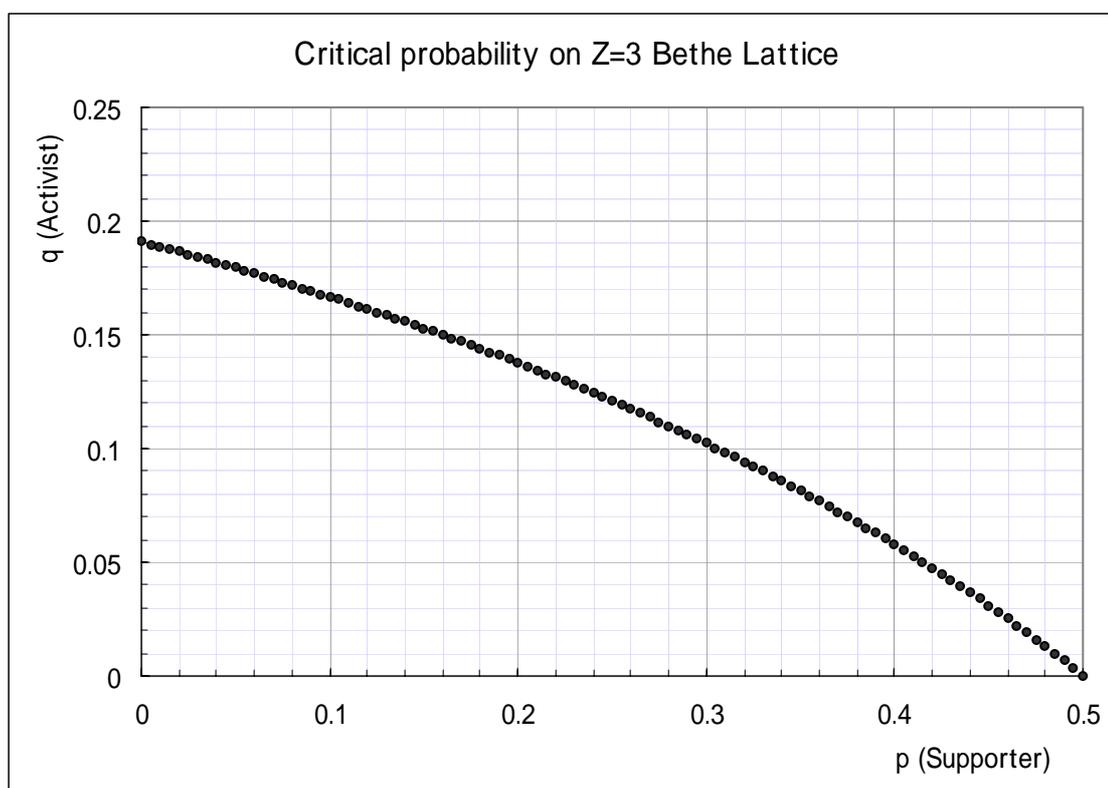
Percolation theory has more merits than the other two theories in that it deals with the mutual relations between activists and supporters and, at the same time, with relations among multiple clusters.

3. Results

In the graph of this model, for example, when activists represent 10% of the population and their supporters' bonds 30%, we have an infinite cluster. But when we have 50% supporters' bond in the society, even without any activists, we also have an infinite cluster. That is the original meaning of threshold P_c in the bond percolation.

And sociologically speaking, it means that supporters of a particular belief come to form a majority in the society. We can say that activists may have only a small effect on the formation of an infinite cluster, but, in a given society, a population of supporters amounting to 30% is large enough even though it falls short of a majority, as long as 10 percents of activists are included. Activists recruit supporters and then organize larger clusters. Thus, they (supporters and activists) will be able to behave as a majority. This shows the importance of the role of activists

The graph is slightly convex upward, which shows that the role of activists is not so important, since so many supporters surround activists and the number of new supporters resulting from persuasion by activists is relatively small. That is why the curve of the graph is convex upward.



4. Discussions

In the graph, for example, we have the value of $(p, q) = (30\%, 10\%)$, where an infinite cluster emerges. At first glance, this probability does not appear to be so large. But the cluster of supporters is large enough that in the course of interaction between both activists and supporters, they are able to work at will between both ends. This precisely replicates the situation in which activists sharing a strong belief can work

efficiently and effectively even though their numbers, as well as the numbers of their supporters, are not sufficient to form a majority. First, activists increase the number of supporters quantitatively, then, changes the quality of the cluster from the finite to the infinite. Finally, the infinite cluster helps increase the qualitative efficacy of both activists and themselves. Both activists and supporters with a strong belief can acquire political power. Here, the effect of the infinite cluster on both activists and supporters should be considered. Activists and supporters affect each other not only quantitatively but also qualitatively within the infinite cluster. The impact of activists on the growth of the cluster comes first, and after the cluster becomes infinite, activists and supporters can work dynamically with the help of the infinite cluster.

Active terrorists and their negative supporters, or environmental activists and their large numbers of supporters provide good examples of these models. Supporters need activists to increase their numbers, and activists need the help of supporters to work effectively.

We should take notice of two things here. First, this example demonstrates how a new order can emerge through the interrelation between activists and supporters. Even a minute amount of help from activists can transform a finite group of supporters into an infinite one. Groups with activists have many catalysts. They help bring about drastic change among supporters not only quantitatively, but also qualitatively. And it is this qualitative change that we should refer to as the emergence of a new order.

Secondly, because of the emergence of this new order, a centralized institution is no longer necessarily needed. Some bigger and stronger clusters may coexist with each other before combining. In a small town, a couple of local environmental groups are sufficient to accomplish good work. They may form a loose, flexible network between themselves. In a small country, active terrorists can be funded and may plan some terrorist attacks through the help of an infinite cluster of supporters. They can easily go beyond their national boundaries to enlarge their external support. Their flexibility and dynamism should be regarded as remarkable.

5. Further discussions

In this article, we have shown the difference between the methods of social science and those of natural science (Newman & Park 2003). Human societies differ from natural groups in two important ways. Firstly, the relations between groups in human society are more diverse, including mutualism. Here, activists and supporters exist as two different groups, both of which have their own roles. That is to say, there is a division of labor. Further, they help each other. Of course, some kinds of mutualism are

found in nature, but, we think, human relations are more complex.

Secondly, there are many clusters, shown here in these models that should be connected with each other. This connectivity is peculiar to humans. Groups are not only made of individuals, but also of subgroups. There are many subgroups. Some are large and some are small. They are not homogeneous in size. They make complex networks and then large groups. These groups thus grow larger within human societies. After their size exceeds the threshold, they become a majority, and may influence the entire society. This dynamic process has been demonstrated through percolation theory.

Indeed, I have to admit that this model is too simple. I know that the reality, especially in society, is more intricate, but in my opinion, no model can be rich enough to fully reflect reality. Of course, we can create much more complex models, but reality will always be far more complex than these. However rich the model may be, it cannot catch up with the complexity of the reality it reflects. What we need, then, is a strategy for grasping reality through the use of our model. When we generate ever more complicated models for the sake of complexity itself, we may falsely assume that such models are complicated enough to fully express reality. That is the weakest point of such complex models.

Simple is beautiful. We agree with R. Axelrod and G. Deffuant on that point (Axelrod 1997, Deffuant et al. 2002, Deffuant et al. 2003). Axelrod says models should be simple, even though the resulting simulated results should be complex. This is the KISS (“Keep it simple, stupid”) principle. G. Deffuant et al. show that making complicated models is very easy, but the study of simple approximations is a good strategy for progress. Philosophical arguments on the relations between reality and models are needed to inform our approach to this strategy. In this case, changes in the size of a cluster, or the change from a finite cluster to an infinite one is our interest, and that is suggestive of the results we would like to see.

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