Evolution of Money Distribution in a Simple Economic Model

Xiang San Liang[†] and **Thomas J. Carter**[‡]
[†]Division of Engineering and Applied Sciences
Harvard University
Cambridge, MA
liang@deas.harvard.edu
[‡]Department of Computer Science
California State University - Stanislaus, Turlock, CA
tom@astarte.csustan.edu

An analytical approach is utilized to study the money evolution in a simple agent-based economic model, where every agent randomly selects someone else and gives the target one dollar unless he runs out of money. (No one is allowed to go into debt.) If originally no agent is in poverty, for most of time the economy is found to be dominated by a Gaussian money distribution, with a fixed mean and an increasing variance proportional to time. This structure begins to be drifted toward the left when the tail of the Gaussian hits the left boundary, and the drift becomes faster and faster, until a steady state is reached. The steady state generally follows the Boltzmann-Gibbs distribution, except for the points around the origin. Our result shows that, the pdf for the utterly destitute is only half of that predicted by the Boltzmann solution. An implication of this is that the economic structure may be improved through manipulating transaction rules.

1 Introduction

Recently there is a surge of interest in applying statistical mechanics laws to study economics[1]. This helps shed light on some aspect of economic phenomena which would otherwise be difficult to see. It has been shown that, for a closed economic system, the equilibrium money distribution often obeys a Boltzmann-Gibbs distribution[2]. This remarkable result follows from an analogy of atoms to agents and energy to money.

Use of the physical equilibrium metaphor, however, should be cautioned. The "invisible force" underlying the market need not be equivalent to a mechanical one, as interactions between agents are due to human activities which are accompanied with unpredictable arbitrariness. As a result, the law of Boltzmann-Gibbs distribution may not be as universal as it appears in physics[1][3][4]. An unfavorable rule of transaction could have it broken. In this paper, we want to use a concrete example to give this problem an illustration.

On the other hand, an economic model mirroring reality faithfully is rarely in equilibrium. In an agent-based model, Carter[5] showed that the time scale for its money distribution to reach a quasi-stationary state is grindingly long. More often than not, observed in the economy might be some distributions on evolution, instead of a final equilibrium. In this sense, transient money states are equally of importance, and more is needed than just the Boltzmann solution.

These two problems, as far as we know, are still not well studied. Particularly lack is a study from analytical point of view, though Monte Carlo simulations with specific problems are not uncommon[1][3]. In this paper, we use an analytical approach to re-examine these problems. We want to demonstrate the applicability limits for the statistical physics laws and, particularly, address through this study the fundamental issue that market rules which may have the Boltzmann distribution assumption violated could be on the other hand utilized to manipulate the economy into a healthier structure. The whole study is limited within the framework of a simple model, which is introduced in the following section. We first derive for this model a master equation, and its corresponding boundary conditions. Asymptotic solution is then sought, and the result analyzed. In Sec. 4.2, we give a description of the transient states, and show how the money distribution evolves toward its equilibrium. Sec. 4.3 explores the steady state of the probability density function, particularly its structure near the origin. These analyses are then verified in a numerical solution (Sec. 5). This work is summarized in Sec. 6.

2 Master equation

The simple model we are about to use in this study is from Carter[5]. At each time step, every agent randomly selects someone else among a collection of agents and gives the target one dollar. If an agent has no money, the agent waits until someone gives him a dollar (no one is allowed to go into debt.) For convenience, suppose that the amount given out by every agent at a transaction is Δx, in some scaled units. (From now on, money is dimensionless). Note that Everyone has to meet someone, but he is not necessarily met by anyone else. One agent could be visited by many agents. For easy reference, we introduce the following notations:

- N: number of total agents
- M: total money (in some scaled units)
- $n_x(t)$: expected number of agents with money x at time t. Agents in this group hereafter will be referred to as x-agents.

Now consider the increase of $n_x(t)$ during a short interval $[t,\, t + \Delta t]$. In the following formulation, the fact that everybody has to visit someone at a time has been used (with probability 1). Lying at the heart of this problem is therefore how one in group $n_x(t)$ is visited by others.

Related to the change of $n_x(t)$ at time t are the following events:

- E_{n_x} = .NOT. $\{x$-agent visited only by one nonzero-agent$\}$
- $E_{n_{x+\Delta x}} = \{(x + \Delta x)$-agent visited by zero nonzero-agent$\}$
- $E_{n_{x-\Delta x}} = \{(x - \Delta x)$-agent visited by two nonzero-agents$\}$
- $E_{n_{x-2\Delta x}} = \{(x - 2\Delta x)$-agent visited by three nonzero-agents$\}$
- ...
- $E_{n_{\Delta x}} = \{\Delta x$-agent visited by $\frac{x}{\Delta x}$ nonzero-agents$\}$
- $E_{n_0} = \{$zero-agent visited by $\frac{x}{\Delta x}$ nonzero-agents$\}$

The increase in expected number of x-agents are then

$$
\begin{aligned}
n_x(t+\Delta t) - n_x(t) = & -n_x \times P(E_{n_x}) + n_{x+\Delta x} \times P(E_{n_{x+\Delta x}}) \\
& + n_{x-\Delta x} \times P(E_{n_{x-\Delta x}}) + n_{x-2\Delta x} \times P(E_{n_{x-2\Delta x}}) + ... \\
& + n_{\Delta x} \times P(E_{n_{\Delta x}}) + n_0 \times P(E_{n_0}).
\end{aligned} \tag{1}
$$

Let $q = \frac{1}{N-1}$, and denote $P_k = P(\text{an agent visited by } k \text{ nonzero-agents})$, which is a binomial for $k \leq N - n_0 - 1$ and zero otherwise,

$$
P_k = \begin{cases} C_{N-n_0-1}^k q^k (1-q)^{N-n_0-1-k}, & \text{for } k \leq N - n_0 - 1 \\ 0, & \text{else} \end{cases} \tag{2}
$$

then the probabilities in (1) can be evaluated as:

$$
\begin{aligned}
& P(E_{n_{x+\Delta x}}) = P_0, \quad P(E_{n_x}) = 1 - P_1, \quad P(E_{n_{x-\Delta x}}) = P_2, \quad P(E_{n_{x-2\Delta x}}) = P_3, \\
& \quad ... \qquad\qquad P(E_{n_{\Delta x}}) = P_{x/\Delta x}, \quad P(E_{n_0}) = P_{x/\Delta x}.
\end{aligned}
$$

So Eq. (1) can be rewritten as

$$
\begin{aligned}
& n_x(t+\Delta t) - n_x(t) \\
& \quad = -n_x(1 - P_1) + n_{x+\Delta x} P_0 + n_{x-\Delta x} P_2 + n_{x-2\Delta x} P_3 + ... + n_{\Delta x} P_{x/\Delta x} + n_0 P_{x/\Delta x} \\
& \quad = -n_x + \sum_{k=0}^{x/\Delta x} n_{x+(1-k)\Delta x} P_k + n_0 P_{x/\Delta x}
\end{aligned} \tag{3}
$$

Normalizing n_x with the total number of agents N, we obtain the probability assigned to money x. For analytical convenience, we may approximately understand it to be $\Delta F(x \leq \text{money} < x + \Delta x)$, the probability associated with the interval $[x, x + \Delta)$. ΔF divided by Δx is the probability density function (pdf) f. We hence obtain an equation governing the evolution of the pdf of money x at time t:

$$
f(x, t + \Delta t) = \sum_{k=0}^{x/\Delta x} f(x + (1-k)\Delta x, t) \; P_k + f(0, t) \; P_{x/\Delta x}, \qquad x \in [0, M]. \tag{4}
$$

This is similar to Einstein's master equation for diffusion[7], but with an expression much more complicated.

In the above formulation, the total number of agents (N) and the total money (M) are preserved. In terms of f, these two conservation laws are expressed as follows:

$$
\text{(a)} \quad \int_0^M f(x,t) \; dx = 1, \quad \text{(conservation of } N) \tag{5}
$$

$$
\text{(b)} \quad \int_0^M x f(x,t) \; dx = \frac{M}{N}. \quad \text{(conservation of } M) \tag{6}
$$

Note (a) and (b) are not constraints in addition to the above formulation. Rather, they are two properties inherently associated Eq. (4).

3 Boundary conditions

The probability space of (4) is not the whole real line. Given a time, f varies on a closed domain $[0,\ M]$. We therefore need to find boundary conditions for the equation at $x = 0$ and $x = M$.

At the left boundary $x = 0$, the change in number of zero-agents at time t, $n_0(t)$, is due to

(a) $E_{\Delta x} = \{\Delta x\text{-agent not visited by nonzero-agents}\}$,

(b) $E_0 = \{\text{zero-agent visited by nonzero-agents}\}$.

Of these two events, (a) is to increase n_0, while (b) is to decrease the number. Following a similar procedure as above for the master equation,

$$\begin{aligned} n_0(t+\Delta t) - n_0(t) &= n_{\Delta x} P(E_{\Delta x}) - n_0 P(E_0) \\ &= -n_0[1-(1-q)^{N-n_0-1}] + n_{\Delta x}(1-q)^{N-n_0-1}, \end{aligned} \tag{7}$$

which reduces to

$$f(0, t+\Delta t) = [f(0,t) + f(\Delta x, t)]\,(1-q)^{N-n_0-1}, \tag{8}$$

if normalized by N and divided by Δx. Recall that $n_0 = N\Delta x f(0,t)$ and $q = \frac{1}{N-1}$, implying that

$$(1-q)^{N-n_0-1} = (1-q)^{1/q - Nf\Delta x} \to e^{-1+f\Delta x}, \qquad \text{as } q \to 0.$$

The left boundary condition then becomes

$$f(0, t+\Delta t) = [f(0,t) + f(\Delta x, t)]\, e^{-1+f(0,t)\Delta x} \tag{9}$$

At the right boundary, $x = M$, f can only have two choices: $\frac{1}{N\Delta x}$ and 0. The former is unstable, as one transaction will revoke the membership of the M-agent. We may hence safely claim that $f(M,t) = 0$.

4 Asymptotic solution

4.1 Simplified governing equation

The master equation (4) can be simplified in the limit of $\Delta t \to 0$, $\Delta x \to 0$. Making Taylor's expansion on both sides, and retaining respectively terms up to $O(\Delta t)$ and $O\left((\Delta x)^2\right)$, we get

$$f + \frac{\partial f}{\partial t}\Delta t = \sum_{k=0}^{x/\Delta x} P_k \left[f + \frac{\partial f}{\partial x}(1-k)\Delta x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,(1-k)^2(\Delta x)^2 \right] + f(0,t)\, P_{x/\Delta x} + \ h.o.t. \tag{10}$$

The key to the problem now is to evaluate the sum over k.

Suppose for the time being x is away from the origin far enough such that $P_{x/\Delta x}$ is ignorable to some given order. This is not a strict constraint. In fact, from Eq. (2),

$$P_k < C_N^k \frac{1}{N^k} < 10^{-6}, \qquad \text{for } k > 10, \text{ when } N = 10000.$$

Therefore, the summation limit $k = x/\Delta x$ may be replaced by $k = N - n_0 - 1$ up to a reasonable precision for x on most of its definition domain, viz.

$$f + \frac{\partial f}{\partial t}\Delta t \approx f \sum_{k=0}^{N-1-n_0} P_k + \frac{\partial f}{\partial x}\Delta x \sum_{k=0}^{N-1-n_0} (1-k)P_k + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 \sum_{k=0}^{N-1-n_0} (1-k)^2 P_k. \qquad (11)$$

We know from binomial distribution $\sum_{k=0}^{N-1-n_0} P_k = 1$, and[8]

$$\sum_{k=0}^{N-1-n_0} kP_k = (N-1-n_0)q = 1 - n_0 q, \qquad (12)$$

$$\begin{aligned}\sum_{k=0}^{N-1-n_0} k^2 P_k &= (N-1-n_0)q + (N-1-n_0)(N-2-n_0)q^2 \\ &= (1-n_0 q) + (1-n_0 q)[1-(n_0+1)q]. \end{aligned} \qquad (13)$$

Eq. (11) then can be simplified as

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} n_0 q \frac{\Delta x}{\Delta t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\frac{(\Delta x)^2}{\Delta t}\left[1 - (n_0+1)q(1-n_0 q)\right]. \qquad (14)$$

When N is large,

$$n_0 q = \frac{n_0}{N-1} \to \frac{n_0}{N} = f(0,t)\Delta x,$$

the first term on the right hand side of (14) is of order $O[(\Delta x)^2]$. The second term becomes

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + O[(\Delta x)^3].$$

These two terms are thence balanced on the order of $(\Delta x)^2$ as $\Delta x \to 0$. Write $\frac{(\Delta x)^2}{\Delta t}$ as μ. The above equation is reduced to

$$\frac{\partial f}{\partial t} = \mu f_0 \frac{\partial f}{\partial x} + \frac{1}{2}\mu \frac{\partial^2 f}{\partial x^2}, \qquad (15)$$

where f_0 is the shorthand for $f(0,t)$. This Fokker-Planck-like equation is similar to the nonlinear Boltzmann equation[6] used in [1], but with different coefficients.

4.2 Transient states

As mentioned in the introduction, generally it takes a long time for the money to reach its steady distribution. In other words, this economy is rarely in equilibrium. Transient states are therefore of very much importance. In this section, we investigate how the economy is evolved toward its equilibrium, and how long the evolution takes.

Eq. (15) is an advection-diffusion equation. When the collection of zero-agents is not empty, $f_0 > 0$, the advection is thence toward the left. If, by any chance, the money distribution is away from the origin, the advection halts, and the solution of f adopts a form of Gaussian. Gaussian dominates the pdf evolution before it hits the left boundary. We will see later in a numerical solution, a non-Gaussian distribution will be adjusted into a normal form very quickly.

During the evolution, the center of the Gaussian keeps fixed if $n_0 = 0$. This is guaranteed by property (6) of the master equation. But the variance increases in proportion to t, and sooner or later, the tail of the Gaussian will hit the left boundary at $x = 0$. It is at this time the whole structure begins to be drifted leftward. As f_0 increases, the drift becomes faster and faster, and eventually the Gaussian gives itself away to other structures. The whole evolution scenario is schematized in Fig. 1.

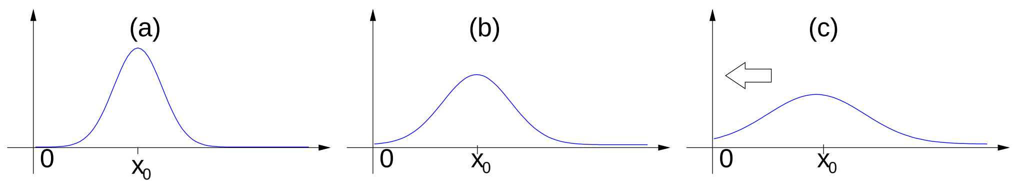

Figure 1: A schematic of the transient states of the pdf evolution: Initially a Gaussian centered at x_0 (a) will maintain its form centered at the same location (b), until its left tail hits $x = 0$. Beginning this time, the whole structure is drifted toward the left (c).

We may also estimate the time scale of duration in maintaining f in the shape of Gaussian. From Eq. (15), Gaussian form is destroyed when its tail hits the boundary. Let the center sit at x_0, then

$$t \sim \frac{2x_0^2}{\mu} = \frac{2x_0^2}{(\Delta x)^2}\Delta t,$$

i.e., the time scale is of the order of $2\left(\frac{x_0}{\Delta x}\right)^2$ times the time step Δt. When x_0 is not too small, this scale is usually very large. Therefore, for most of time, the economy is dominated by a Gaussian money distribution. In other words, the economy chooses for the money distribution a Gaussian on its route of evolution toward the equilibrium, which we are now to explore.

4.3 Steady state

In the steady state, $\partial/\partial t = 0$, Eq. (15) is simplified to an ordinary differential equation

$$\frac{\partial^2 f}{\partial x^2} + 2f_0\frac{\partial f}{\partial x} = 0. \tag{16}$$

Notice that here $f_0 = f(0, \infty)$ is a constant in x. The ODE thus can be easily solved, with a solution in the form

$$f = De^{-2f_0 x}. \tag{17}$$

Constants D and f_0 can be determined by the two conservation properties. When M is large, they are

$$D = 2f_0 \tag{18}$$

$$f_0 = \frac{N}{2M}. \tag{19}$$

The equilibrium solution is therefore

$$f(x, \infty) = \frac{N}{M}e^{-\frac{N}{M}x}, \tag{20}$$

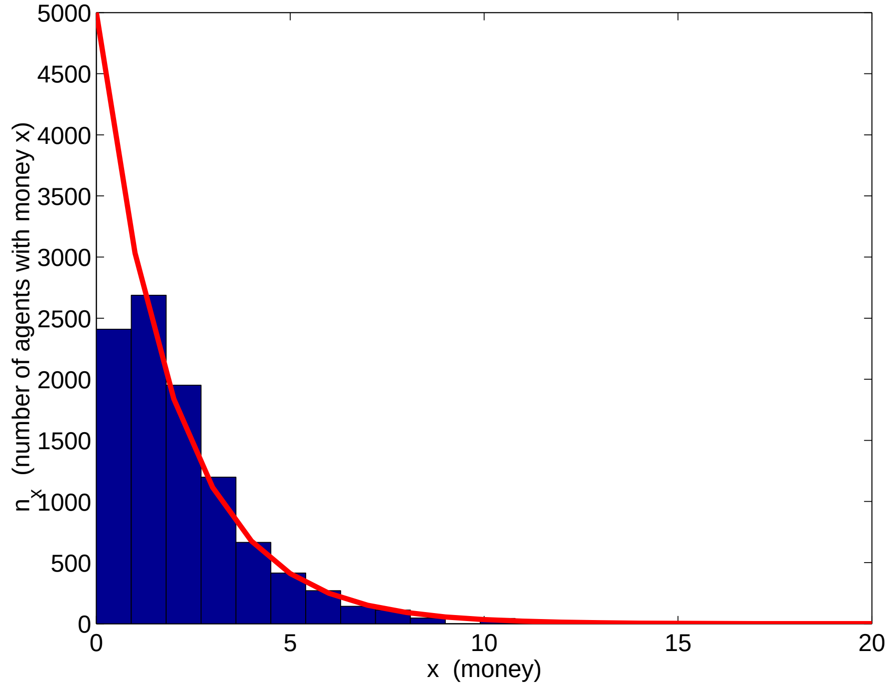

Figure 2: Histogram of a Monte Carlo simulation with $N = 10000$, $M = 20000$, and $\Delta x = 1$. The solid curve is the Boltzmann-Gibbs money distribution for this economy. Our estimation, $n_0 = 2500$, $n_1 = 2793$, agrees well with the simulated result near the origin.

same as the Boltzmann solution obtained from the maximal entropy principle[5].

Note that Eq. (20) does not hold near the origin. We emphasized this when we made the assumption in simplifying the master equation. What is of interest is: the pdf at the origin actually has been determined in (19). It is equal to $\frac{N}{2M}$, half of the pdf predicted by (20), the Boltzmann solution.

We can learn more about the steady probability distribution near the origin through the boundary condition. Using f_k to indicate $f(k\Delta x, \infty)$, the equilibrium form of Eq. (9) is:

$$f_0 = [f_0 + f_1]\, e^{-1+f_0\Delta x} \tag{21}$$

which, when substituted with $\frac{N}{2M}$ for f_0, gives

$$f_1 = \left[e^{1-\frac{N}{2M}\Delta x} - 1\right] \frac{N}{2M}. \tag{22}$$

As a verification, we have run a Monte Carlo simulation with $M = 20000$, $N = 10000$, $\Delta x = 1$. In this economy, our model predicts that the numbers of zero-agents and Δx-agents are, respectively, $n_0 = N \times \frac{N}{2M}\Delta x = 2500$, $n_{\Delta x} = (e^{1-\frac{N}{2M}\Delta x} - 1)n_0 = 2793$. The simulated equilibrium state is shown in the histogram Fig. 2. Shown also in the figure is the Boltzmann solution, which agrees very well with the simulation except for a small region around the origin. Apparently, n_0 and $n_{\Delta x}$ are very close to our predictions. Particularly, n_0 is approximately half of 5000, the pdf at $x = 0$ given by (20).

5 Numerical solution

In order to have a better understanding of the evolution scenario described above, we present in this section a numerical solution for the master equation (4), with $N = 10000$, $M = 956779$, and $\Delta x = 1$. [The value of M is not essential. It takes this number from the initial beta distribution (see below) we generate numerically.] The result is shown in Fig. 3. The initial condition is approximately a beta distribution on $[0, 100]$ (Fig. 3a). From the solution, after only ten time steps, it gets adjusted into a Gaussian-like shape, which is centered at $x_0 = \frac{M}{N} = 96$. This state lasts for a long time, and even after it hits the left boundary at about $t = 2000$, most of it still keeps a norm form. The final steady state is like an exponential decay, as predicted by Eq. (20). For points near the left boundary, (20) is not valid. But the expected numbers of agents at $x = 0, 1$, $n_0 = 52$ and $n_{\Delta x} = 89$, agree well with the prediction with formulas (19) and (22).

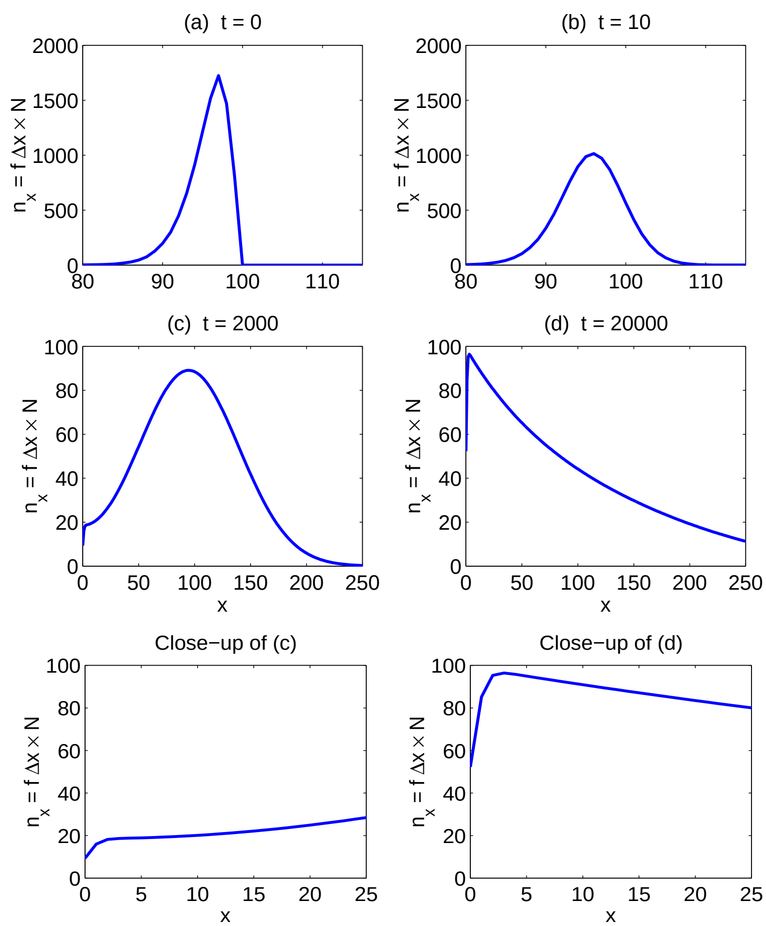

Figure 3: A numerical solution of the master equation (4) with $N = 10000$, $M = 956779$, and $\Delta x = 1$. The value of M is not essential. It is from the initial beta distribution we generate numerically.

6 Summary

With the aid of a master equation, we have investigated the evolution of money distribution in an agent-based simple economic model. If originally no agent is in poverty, for most of time the money is distributed in a Gaussian form, with the mean fixed at x_0, the arithmetic average of money for the economy, and the variance increasing with time. This geometric structure is very stable, lasting for a time scale proportional to $\left(\frac{x_0}{\Delta x}\right)^2$, with $\frac{x_0}{\Delta x}$ being the ratio of the mean to the minimal amount of money exchange per transaction. After hitting the boundary, the structure is drifted leftward, faster and faster, until an equilibrium is reached with a completely different structure.

The equilibrium state of the economy has also been explored. The stationary distribution is roughly the same as the Boltzmann solution obtained from the maximal entropy principle, except for points near the origin. By our estimation, the number of agents without any money is only half of the number predicted for the same group from the Boltzmann solution. This estimation has been verified in a Monte Carlo simulation.

We remark that the solution thus obtained has practical implications, though the economy itself is too simple to model the real world. Given the number of total individuals and the total money, by Eqs. (19) and (22) f_0 is fixed, but f_1 drops with Δx increased. That is to say, in this economy, we may reduce the number of the poor by increasing the amount for one transaction. It is of our interest to explore the possibility of improving the structure of a society through manipulating market rules with a more sophisticated model.

Acknowledgment X. San Liang has benefited from the discussions with Justin Ferris and Olinto Linares-Perdomo.

References

[1] A. Drăgulescu, and V.M. Yakovenko, Eur. Phys. J. B 17, 723 (2000), and references therein.

[2] see, for example, the book by M. Aoki, *New Approaches to Macroeconomic Modeling* (Cambridge University Press, Cambridge, 1996).

[3] A. Chakraborti, and B.K. Chakrabarti, Euro. Phys. J. B 17, 167 (2000).

[4] S. Ispolatov, P.L. Krapivsky, S. Redner, Eur. Phys. J. B 2, 267 (1998).

[5] T. Carter, *An introduction to information theory and entropy*, (CSSS Lecture Notes, The Santa Fe Institute, New Mexico, 2003).

[6] Lifshitz, E.M., and L.P. Pitaevskii, *Physical Kinetics*, (Pergamon Press, New York, 1993).

[7] Einstein, A., *Investigations on the Theory of the Brownian Movement*, (Dover Publications, INC., New York, 1956).

[8] Abramowitz, M., and I.A. Stegun, *Handbook of Mathematical Functions*, (Dover Publications, INC., New York, 1970).