Chapter 1

Level Statistics of Complex Systems

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The dynamical studies of various complex systems require a statistical information about eigenvalues and eigenfunctions of the generators of the dynamics. It is very useful, if possible, to identify a common mathematical structure among them and analyze it to gain information. Our successful search in this direction leads to Calogero-Sutherland Hamiltonian, a one-dimensional quantum Hamiltonian with inverse-square interaction, as the common base. This is because both, the eigenvalues of complex generators, and, a general state of Calogero Hamiltonian, evolve in an analogous way for arbitrary initial conditions. The varying nature of the complexity is reflected in different form of the evolution parameter in each case. A complete investigation of Calogero Hamiltonian can then help us in the spectral analysis of complex systems.

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The complexity of interactions seems to be a generic feature of almost all physical systems in nature. The lack of detailed information about complicated interactions leaves a degree of uncertainty in the exact determination of the generators of system dynamics. The matrix elements of the generators, as a result, behave like random variables and their statistical properties can be studied by using the tools of Random Matrix Theory (RMT) [1, 2, 3, 4]. ..

Many properties of a physical system can theoretically be formulated in terms of the eigenvalues and eigenfunctions of the generator of its dynamics. However,

due to randomness and uncertainty associated with complicated interactions, a useful information can be extracted only from by statistical analysis of the properties. The RMT was originally introduced to understand the statistical behavior of complex quantum systems with dynamics governed by a Schrodinger equation [1, 2, 3]. Later on, it was shown to be applicable to the physical properties of other wave equations too, for example, electromagnetic waves in a microwave cavity, elastomechanics, signals in a brain, share market fluctuations etc. [3]. The old RMT, known as Wigner-Dyson theory, has been very successful as a model of complex generators with wavefunctions extended in entire system or with a coherent scattering of waves. In other words, the interaction between various parts of the system, as well as uncertainty associated with them, is almost same; the distribution of various matrix elements is therefore of same type and independent of the choice of the basis. However properties of the systems with localized wave dynamics can not be modeled by Wigner-Dyson theory. This is because the presence of localized waves in the system is an indicator of non-homogeneous interactions between its various parts. A matrix elements describes the interaction of one basis state to another (or one part of the system to other) through the operator. Thus a varying degree of interaction between various basis states results in different strengths of various matrix elements with some of them correlated. Unfortunately not much information about such ensembles has been available so far. Our objective in this study is to suggest a way to fill in this information gap.

A generic random matrix can be described as a matrix with some (or all) of its elements as randomly distributed. Here the distributions of various matrix elements need not be same, may or may not be correlated and some of them can be non-random too. One of the most simple, however very useful, class of random matrix ensembles is known as the standard Gaussian ensemble (or Wigner-Dyson ensemble) with matrix elements distribution given by $P(H) \propto \mathrm{e}^{-\mathrm{Tr}H^2}$ [4]. These are least information or maximum entropy ensembles under the single constraint $< \mathrm{Tr}H^2 >= constant$, with $<>$ implying the ensemble average; their statistical properties depend only on the underlying symmetry conditions and are independent of the specific details of the ensemble. Note various matrix elements in this case are independently distributed. The main universality classes of standard Gaussian ensembles (SGE), are based on the exact symmetry conditions such as time-reversal, angular momentum and chiral symmetry. A random perturbation of SGEs e.g. due to partial violation of one of the symmetries subjects the ensemble to undergo a transition from one universality class to another. The intermediate states of the transition, known as Brownian ensembles, are characterized by strength of the perturbation as well as by the universality classes of the two end-points [5, 4]. The connection of Brownian ensembles with non-stationary states of Calogero Hamiltonian is already well-known [4]. This follows because the equation governing the evolution of eigenvalues during the transition can be transformed to a Schrodinger equation governed by a Calogero-Sutherland Hamiltonian; the perturbation parameter in the former plays the role of time in the latter. This analogy is very useful

in studying the properties complex systems with partially broken symmetries, e.g. atoms, molecules or quantum dots etc. in a magnetic field, which can be well-modeled by Brownian ensembles [4, 3, 1, 2].

The complexity in a variety of physical systems leads to localization of wave-dynamics [3]. Under certain conditions or limits, many of these cases, for example, Anderson Hamiltonian, spin glass etc, can be modeled by Gaussian ensemble in which various matrix elements, although distributed with different variances and means, are uncorrelated [1, 2, 7, 8]; some of them may have zero variance and therefore can be non-random too. As shown by a recent study [7], the statistical properties of a multi-parametric Gaussian ensemble with uncorrelated elements can also be mapped to particle-statistics of the Calogero-Sutherland Hamiltonian. The mapping is achieved by first defining a complexity parameter for the ensemble which is basically a function of the correlation strengths between various matrix elements. The evolution of eigenvalues with changing complexity parameter is then shown to be governed by a diffusion equation which can be reduced to the Schrodinger equation for Calogero-Sutherland Hamiltonian. This results in equivalence of the eigenvalue distribution for the uncorrelated ensemble at a given value of complexity parameter to the non-stationary state of Calogero-Sutherland Hamiltonian at a given time. is therefore identified with time. In this paper, we adopt the same technique to establish the connection between correlated Gaussian ensembles and Calogero-Sutherland Hamiltonian. The advantages of this analogy are manifold: (i) a lot of information is already available about particle correlation which can be directly be used for the eigenvalue correlations of correlated ensembles, (ii) the appearance of Calogero-Sutherland Hamiltonian in a wide range of complex phenomenon is already known. The revelation of this new connection to correlated Gaussian ensembles not only connects the latter to other complex phenomenon, it also suggests a hidden structure underlying the complex world.

A Gaussian ensemble of Hermitian matrices with correlated elements can be described by a matrix elements distribution

$$\tilde{\rho}(H,b) = C\exp[-\sum_{s=1}^{2}\sum_{k,l;k\leq l}\sum_{i,j;i\leq j} b_{ijkl;s}H_{kl;s}H_{ij;s})] = C\rho(H,b) \qquad (1.1)$$

with C as the normalization constant and b as the set of coefficients $b_{ijkl;s}$ (note $b_{ijkl;s} = b_{klij;s}$). Here similar pairs are counted only once in the summation over all possible pairs of indices $\{i,j\}$ and $\{k,l\}$. The subscript s to a variable refers to one of its components with β as their total number: $H_{kl} = \sum_{s=1}^{\beta}(i)^{s-1}H_{kl;s}$. The parameter β contains the information about underlying symmetry of the system. For systems with time-reversal symmetry and integer angular momentum, the Hamiltonian H in a generic representation is a real-symmetric matrix and therefore $\beta = 1$. The Hamiltonians for system without time reversal symmetry are, in general, complex Hermitian which gives $\beta = 2$.

The distribution parameters $b_{ijkl;s,s'}$ are the measures of correlations between pairs of the matrix elements: $< H_{ij;s}H_{kl;s'} >= \frac{\partial \log C}{\partial b_{ijkl;s,s'}}$. In general, different

system conditions can give rise to different sets of distribution parameters b. A slight perturbation of the complex region due to a change in system parameters perturbs the matrix elements H_{kl} and therefore the probability density $\rho(H,b)$. The latter being a function of both H and b, the derivatives of ρ with respect to matrix elements can be expressed in terms of the parametric derivatives. This can be explained as follows:

$$\frac{\partial \rho}{\partial H_{kl;s}} = -2 \sum_{i,j;i\leq j} b_{ijkl;s} H_{ij;s} \rho \tag{1.2}$$

$$\frac{\partial \rho}{\partial b_{ijkl;s}} = -\frac{2}{G_{ijkl}} H_{kl;s} H_{ij;s} \rho \tag{1.3}$$

with $G_{ijkl} = 1$ for $\{ij\} \not\equiv \{kl\}$, $G_{ijkl} = 2$ for $\{ij\} \equiv \{kl\}$. By using eq.(2) and (3), a drift in the matrix space can be written in terms of a drift in the parametric space,

$$\begin{aligned} \sum_{kl;s} H_{kl;s} \frac{\partial \rho}{\partial H_{kl;s}} &= -2 \sum_{i,j,k,l;s} b_{klij;s} \; H_{ij;s'} \; H_{kl;s} \rho \\ &= \sum_{i,j,k,l;s,s'} G_{klij;s} \; b_{klij;s} \frac{\partial \rho}{\partial b_{klij;s}} \end{aligned} \tag{1.4}$$

Similarly a diffusion in the matrix space can be expressed as a combination of drifts in the parametric space,

$$\frac{\partial^2 \rho}{\partial H_{kl;s}^2} = -2 b_{klkl;s} + 4 \sum_{ij} \sum_{mn} b_{klij;s} \; b_{klmn;s} \; H_{ij;s} \; H_{mn;s} \tag{1.5}$$

$$= -2 b_{klkl;s} + 2 \sum_{i,j,m,n} G_{ijmn} \; b_{klij;s} \; b_{klmn;s} \frac{\partial \rho}{\partial b_{ijmn;s'}} \tag{1.6}$$

now by combining the equalities (4) and (6), one can write

$$\sum_{k,l;s} \frac{\partial}{\partial H_{kl;s}} \left[\frac{1}{2} \frac{\partial \rho}{\partial H_{kl;s}} + \gamma H_{kl;s} \; \rho \right] = \sum_{i,j,kl;s} G_{ijkl} \; f_{ijkl;s} \; \frac{\partial \rho}{\partial b_{ijkl;s}} + C_1 \rho \tag{1.7}$$

with $f_{ijkl;s} = \gamma b_{ijkl;s} - \sum_{m,n} \; b_{ijmn;s} \; b_{mnkl;s}$ and $C_1 = -2 \sum_{kl;s} b_{klkl;s}$. Here the parameter γ is arbitrary and marks the end of the transition.

The eq.(7) appears complicated, with many parametric derivatives present on its right side. However it can be simplified by using a transformation of variables in the parametric space. Similar to the case of uncorrelated, multi-parametric Gaussian ensembles [7], it is possible to define a transformation from the set $\{b_m\}$ to another set $\{y_n\}$, $m, n = 1 \rightarrow M$ (with $M = \beta N(N + 2 - \beta)/2$), which satisfies the following condition:

$$\sum_{i,j,kl;s} G_{ijkl}\ f_{ijkl;s}\ \frac{\partial \rho}{\partial b_{ijkl;s}} = \frac{\partial \rho}{\partial y_1} \tag{1.8}$$

The condition implies that parameters y_j, $j > 1$ behave as constants of dynamics with density $\rho(H)$ undergoing a single parametric evolution in the matrix space

$$\sum_{k,l;s} \frac{\partial}{\partial H_{kl;s}} \left[\frac{\partial \tilde{\rho}}{\partial H_{kl;s}} + \gamma H_{kl;s}\ \tilde{\rho} \right] \quad = \quad \frac{\partial \tilde{\rho}}{\partial y_1} \tag{1.9}$$

where the last term on right side of eq.(7), namely, $C_1\rho$ is absorbed in the normalization constant of $\tilde{\rho}$. where y_1 is a function of various parameters $b_{ijkl;s}$ and can therefore be termed as the complexity parameter,

$$Y = \frac{1}{\tilde{M}} \sum_{i,j,k,l;s} \int \frac{\mathrm{d}b_{ijkl;s}}{Z} + constant \tag{1.10}$$

where $Z = \sum_{ijkl;s} G_{ijkl}\ b_{ijkl;s}$ and $\tilde{M}$ as the number of nonzero elements in set b.

The parametric space transformation $b \to y$ maps the probability density $\rho(H, b)$ to $\rho(H, y(b))$. As a result, ρ depends on various parameters y_j, $j = 1 \to M$. However, the diffusion in the matrix-space H is governed by y_1 only; the rest of them, namely, y_j, $j > 1$ remain constant during the evolution. Note it is always possible to define a transformation from the set $b \to y$ with y_j, $j > 1$ as the constants of dynamics generated by L. This can be explained as follows. A matrix element, say H_{ij}, describes how a basis state ψ_i interacts with state ψ_j through H. This results in dependence of the matrix element correlations and, thereby, of the set b, on the basis parameters e.g. basis indices. As the basis remains fixed during the evolution, the suitable functions of basis parameters can be chosen to play the role of y_j, $J > 1$. (Note a similar transformation has been used to obtain the single parametric evolution of multi-parametric Gaussian ensembles of Hermitian matrices; see [7] for details).

An exact diagonalization of eq.(9) results in a diffusion equation for the eigenvalues μ_i, $i = 1 \to N$ of Hamiltonian H. Let

$$P(\mu; y_1) \equiv P(\mu_1, ..\mu_N; y_1) = C \int \rho(H)\ J(H|\mu, U)\ \mathrm{d}U \tag{1.11}$$

be the eigenvalue distribution, with J as the Jacobian of transformation from matrix (H) space to eigenvalue (μ) and eigenfunction (U) space. By using the eigenvalue equation $H\ U = \mu\ U$ and the integration by parts technique, it can be shown that (see [7] for details)

$$\frac{\partial \tilde{P}}{\partial Y} \quad = \quad \sum_n \frac{\partial}{\partial \mu_n} \left(\gamma \mu_n \tilde{P} \right) + \sum_n \frac{\partial}{\partial \mu_n} \left[\frac{\partial}{\partial \mu_n} + \sum_{m \neq n} \frac{\beta}{\mu_m - \mu_n} \right] \tilde{P} \tag{1.12}$$

(with $Y \equiv y_1$, the notation changed for an easy reference). The eq.(12) describes the evolution of the eigenvalues due to variation in Y from an arbitrary initial state, say $P(\mu_0, Y_0)$ occurring at $Y = Y_0$. But the parameters y_j $(j > 1)$ being constants of motion, the initial ensemble should be such that its parameters b give rise to a set of function y_j $(j = 2 \to M)$ same as those for $P(\mu_0, Y_0)$; the condition is satisfied as long as both, the initial ensemble as well as the ensemble under consideration, are represented by the same basis.

The evolution of the eigenvalues tends to an steady state in the limit $\frac{\partial P}{\partial Y} \to 0$ or $Y \to \infty$. which occurs when $b_{ijkl;s,s'} \to \gamma \delta_{ik}\delta_{jl}\delta_{ss'}$ and, equivalently, $\rho \to \mathrm{e}^{-(\gamma/2)\mathrm{Tr}H^2}$ (from eq.(1)) i.e. a Standard Gaussian ensemble. As expected, the solution of eq.(12), in the limit, turns out to be a standard Gaussian ensemble too[4]: $P(\mu) = \prod_{i<j} |\mu_i - \mu_j|^\beta \mathrm{e}^{-\sum_k \mu_k^2}$ (thus a GOE for $\beta = 1$ and a GUE for $\beta = 2$). The eq.(12) therefore describes a transition from a given initial ensemble (with $Y = Y_0$) to a standard Gaussian ensemble with $Y - Y_0$ as the transition parameter. The non-equilibrium states of this transition, given by non-zero finite values of $Y - Y_0$, are various correlated Gaussian ensembles corresponding to changing values of parameters b_{kl} which therefore represent changing states of complexity in a system. Note eq.(12) describes the evolution of probability density P from arbitrary initial conditions, say $P(\mu_0, Y_0)$; the distribution $P(\mu, Y) = \int P(\mu, Y|\mu_0, Y_0) P(\mu_0, Y_0) \mathrm{d}\mu_0$ of a given ensemble can therefore be found by solving eq.(12) by using a convenient initial ensemble. Here the "convenience" depends on mathematical tractability of the integrals as well as on involved physics [7].

The determination of the eigenvalue correlations requires a prior knowledge of the joint probability distribution of eiegnvalues $P(\mu, Y)$. For example, the correlation $R_n(\mu_1, ..\mu_n; Y)$ among n eigenvalues at a given parameter value Y can be obtained by integrating $P(\mu, Y)$ over $N - n$ eigenvalues: $R_n = \frac{N!}{(N-n)!} \int P(\mu, Y) \mathrm{d}\mu_{n+1} .. \mathrm{d}\mu_N$. This motivates us to seek a solution of eq.(12). The desired $P(\mu, Y)$ can be obtained by using the connection of eq.(12) to Calogero-Sutherland Hamiltonian. The connection can be shown by applying the transformation $\Psi = P/|Q_N|^{\beta/2}$ to eq.(12) which reduces it in a form (using $\gamma = 1$ for simplification):

$$\frac{\partial \Psi}{\partial Y} = \hat{H} \Psi \tag{1.13}$$

where 'Hamiltonian' $\hat{H}$ turns out to be the Hamiltonian governing the dynamics of N "particles" interacting via an inverse square potential and confined by a quadratic potential in one dimensions

$$\hat{H} = \sum_i \frac{\partial^2}{\partial \mu_i^2} - \frac{1}{2} \sum_{i<j} \frac{\beta(\beta - 2)}{(\mu_i - \mu_j)^2} - \sum_i \mu_i^2 \tag{1.14}$$

As $Y \to \infty$, the particles tend to their ground state $\psi_0 = \prod_{j<k}^N |\mu_\mathbf{i} -$

$\mu_{\mathbf{j}}|e^{-\frac{1}{2}\sum_k |\mu_k|^2}$ with a distribution ψ_0^2; note ψ_0 gives the correct form for $P(Y \to \infty)$.

Following standard quantum mechanics, a non-stationary state ψ of a Schrodinger equation can be expressed as a sum over the eigenvalues λ_k and eigenfunctions ϕ_k, $k = 1 \to N$: $\psi(Y) = \sum_{k>0} \exp[-\lambda_k(Y - Y_0)]\phi_k(\mu)\phi_k^*(\mu)$ with r representing the particle positions. The allows us to express the probability density $P(\mu, Y|\mu_0, Y_0)$ of the eigenvalues of ensembles of matrices H as a sum over the eigenvalues and eigenfunctions of $\hat{H}$:

$$P(\mu, Y|H(Y_0)) = |\frac{Q_N(\mu)}{Q_N(\mu_0)}|^{\beta/2} \sum_{k>0} \exp[-\lambda_k(Y - Y_0)]\zeta_k(\mu)\zeta_k^*(\mu_0) \qquad (1.15)$$

where $\mu_0 \equiv (\mu_{01}, \mu_{02}, ..., \mu_{0N})$ are the position of eigenvalues at $Y = Y_0$. The joint probability distribution $P(\mu, Y)$ can then be obtained, in principle, by integrating over initial ensemble $P(\mu_0, Y_0)$ which further leads to the static eigenvalue correlations R_n [6]. The above correspondence can also be used to map the multi-parametric correlations of levels to multi-time correlations of the particle-positions [7]. A detailed knowledge of the latter therefore can give a great deal of information about the eigenvalues of a wide range of complex systems (see [7] for details).

As well-known, the non-stationary states of a Schrodinger equation are related to each other by time-evolution operator: for example, a state ψ_2 at "time" Y_2 is connected to another state ψ_1 at "time" Y_1 by a unitary operator $U(Y_2, Y_1)$: $\psi_2(Y_2) = U(Y_2, Y_1)\psi_1(Y_1)$. The equivalence between complex systems and Calogero-Sutherland Hamiltonian therefore implies that the eigenvalue distributions of the complex systems corresponding to complexity parameters Y_1 and Y_2 will also be connected. This would also reflect in their physical properties based on spectral fluctuations e.g conductance (assuming existence of ergodicity, that is, ensemble averages same as spectral averages). The mapping therefore hints at a deep-rooted connectivity in the world of complex systems and it should be explored in detail.

In the end, we summarize our main results. We find that, within random matrix framework, a wide range of complex systems can be characterized by a complexity parameter, basically a measure of the strengths of underlying interactions and their uncertainties. The characterization is then used to show the equivalence of the eigenvalue dynamics of a system due to its changing complexity to time-dependent particle dynamics of the Calogero-Sutherland Hamiltonian. The appearance of CS Hamiltonian is not restricted only to the spectral properties; it has been known to manifest itself in other properties of complex systems too [9]. A detailed investigation of Calogero-Sutherland Hamiltonian in arbitrary dimension can therefore give a lot of useful information about variety of complex systems and is very much desirable. ..... .

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