THE INHOMOGENEOUS PRODUCT-POTENTIAL SOCIAL SYSTEMS

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Abstract

The main problem that was presented in this paper is problem of existence of potential fields. For solving this problem was used method of smooth fields on the solid domain and product integrals. For of smooth fields will be write the system of infinitesimal equations (system of partial differential equations) that must be hold for all potential fields and then was found solutions of infinitesimal equations. Then smooth potential field on domain will be transformed into discrete potential marking on embedded graph of relation by using product integrals.

Key words: *social network, balanced groups, group of reactions, potential fields, infinitesimal equations.*

Introduction.

This article contains theory of potential networks as very important type of locally interrupted system with relations and reactions. The potential networks were deduced from locally interrupting system by using so call "principle of maximum non-ergodicity". The system are balanced in social sense if the set of psychological reactions on the graph of relation satisfy the principle of maximum non-ergodicity and this system of reactions are product potential on the so call "two-steps" graph of relations [1]. In real life survive only relatively stable groups and we can observe only stable groups that in social science call balanced. So reason why social system (group) is stable based on hidden potentiality of reactions: only social systems with potential system of reactions (potential fields) are stable (balanced in social sense). This conclusion is not a big surprise for natural (physical) systems. For instance the system consisting of a star and a single planet is stable because the gravitational interaction is potential (friction is absent). The potentiality of gravitational interaction means that work done along any closed path is zero. But for social science and, particularly for human groups, a similar property comes as a surprise. Therefore, the reason why balanced groups (systems) can exist forever is a hidden **potentiality** of human reactions inside balanced groups. This is the main reason this system was studied in this article.

1. Set up of problem and definitions. The general description problem stability of social system and mathematical models reader can find in [1-3].

The social system or network is oriented graph (graph of relations), where any node represents the person and any edge represents relations. If any edge of social system marked by elements of algebraic group G (group of psychological reactions on the type of relations, where g^2 = e) and if product all elements along arbitrary closed path in the order in which the path goes equals unit element, then social system has called the social potential system or network.

The main problem that was solved in this article was problem of the existence of social potential marking (fields). For this purpose was created special method by using the smooth potential fields on a rigid medium. For smooth potential fields we wrote the system of infinitesimal equations that must hold for all potential fields. It is a system of partial differential equations that was transformed into the system of linear equations with one additional condition on the solution: the matrix-solution and field have to be an anti-commutative pair.

Then we found solutions of infinitesimal equation: the solution is any parameterizations of intersection of second degree surface (set of matrices that A^2 =E) and arbitrary plane. The set of solution can be represented in the few canonical forms. Then the property of potentiality was checked by computer calculation for all types of potential fields.

Right now we define the product or path ordered integral for the solid domain.

A path ordered integral for non-Abelian fields (P-integral) can be defined as

$$P\left[\int Adx\right] = lim_{n->\infty}\prod_{i=1}^{n}(A(x_i)\delta x_i)$$

, where the product goes along the path in the order in which the path goes.

The properties of P-integral see Gantmacher [4].

It is easy to see in our case that the definition of the integral depends from what the kind of integer n will be taken. If all number are even we get one result; for odd number we get completely different result: the determinant of A is -1 and the determinant of product the even (odd) number of matrices is +1 (-1). If we want to use P-integral as tool for theory of potential system, the number of step (n) must always be even integer number and we call it P_2 - integral. If number steps n is odd then we will receive P_1 - integral. We will use both.

2. The general description of the field (marking) A(x) and properties of product N matrices.

For a heterogeneous potential system we have to find general description of the field (marking) A(x), where $A(x)^2$ =e, A(x) belongs to group of psychological reactions for all x, and x is vector of parameters and when our group is the two dimensional group of the matrices. It is easy to see that the arbitrary 2 by 2 matrix A satisfy this condition if and only if A= GZG^{-1}, where G is arbitrary element of GL(2) and

$$Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is also easily seen that Z^2=E and A^2= GZG^{-1} GZG^{-1} = GZZG^{-1} = GZG^{-1} = A. For elements a(i,j) of the matrices A= GZG^{-1} the next property hold: a(1,1) = - a(2,1) and a(1,2)a(2,1) = 1 - $a(1,1)^2$. We then to deduce by direct calculation: a(1,1)= (bd -ac)/(ad -bc), a(1,2) = (a^2 - b^2)/(ad -bc), a(2,1) = (d^2 - c^2)/(ad -bc), a(2,2) =(ac -bd)/ (ad -bc), where

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

, where det G = ad - bc not equal zero.

This means that the set of 2-dimensional matrices that its square is unit matrix is a two dimensional manifold in three dimensional space D={ A(a,b,c)= { $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where bc = 1- a^2 }.

4. The infinitesimal equations for social potential fields and solution of the existence problem for general two-dimensional social potential system on the solid set.

Let us define set of 2 by 2 matrices A(a,b,c) that square equal E as $A(a,b,s) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where bc= 1- a^2.

Then take the square [0,1]x[0,1] on the plane xOy. Let h=1/n, where n is integer number,and divide the square on the n^2 small squares with length of side equal h. Then divide any small square on the two triangles by main diagonal. Suppose we three smooth functions f_1(x,y),f_2(x,y), and f_3(x,y) defined on the square [0,1]x[0,1], where f_2(x,y)f_3(x,y) = 1 - $f_1(x,y)^2$. In this case we automatically define set of matrices A(f_1(x,y), f_2(x,y), f_3(x,y)) on the square [0,1]x[0,1]. Let us take the triangle grid with step h and define the network of mark of edges by matrices A(f_1(x,y), f_2(x,y), f_3(x,y)) taken in middle points of edge. We can start from interval y=0 and 0 $\leq$ x $\leq$ 1. We will find condition when heterogeneous (nonhomogeneous) distribution is potential. It means that product integrals along two curves started in same point and ended in same point are equal. It means that we have to find condition on the f_1(x,y), f_2(x,y), f_3(x,y): M(c_1(C,D), A(f_1(x,y), f_2(x,y), f_3(x,y))) = M(c_2(C,D), A(f_1(x,y), f_2(x,y), f_3(x,y))) , where C and D are arbitrary points on the square [0,1]x[0,1] and c1 and c2 are curves connected C and D.

Lemma. The social fields with values in the set of matrices A(f_1(x,y), f_2(x,y), f_3(x,y)) is potential if and only if satisfy the system of differential equations) :

$$A(d^2A/dxdy) + dA/dydA/dx = 0$$

, where A^2 = A and f_2(x,y)f_3(x,y) = 1 - $f_1(x,y)^2$.

The social potential marking (field) have to be solution to nonlinear partial differential equation AA_{xy} + A_yA_x =0 or A_{xy} A+ A_yA_x =0. The equation is actually the system of linear ODE: $(AA_x)_y$ = A A_{xy}+ A_yA_x and $(AA_y)_x$ = AA_{xy} + A_xA_y. So our equations are

$$(A_xA)_y = 0 \, and \, (AA_x)_y = 0.$$

The last equations mean that

$$A_xA = B(x) \, and \, AA_y = D(y),$$

where B(x) (D(y)) is matrix-function only from one variable x (y) and equality AB = -BA (AD = - DA) hold.

5. General and particular solutions of 2-dimencional infinitesimal equation.

The system of differential equations (infinitesimal condition for potentiality) AA_{xy} +A_yA_x = $(A_xA)_y$=0 and $(AA_{y)})_x$ =0 are really is first order system of ODE.

What follow is the general description of all non-constant solution. We will start from AA_y = C(y),where C(y) is matrix-function only from one variable y and equality AC = - CA hold. So first of all we have to describe all matrices C that AC = - CA. Really we will describe a set of A(a,b,c) where AC = - CA: the (a, b, c) must satisfy equation 2aC_1 + cC_2 + bC_3=0 for any matrix

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & -C_1 \end{pmatrix}.$$

It means that all triplet number C_1, C_2, and C_3 we can find solution by solve system of algebraic equations: a^2 + bc =1 and 2aC_1 + cC_2 + bC_3 =0.

It means that (a, b, c) have to satisfy quadric equation: $c^2(C_2^2$ + 2bc(C_2C_3 + $2C_1^2$) + $b^2C_3^2$ = $4C_1^2$.

We can then prove by direct calculation the next theorem.

Theorem. For any triplets (C_1(y), C_2(y), C_3(y)) there are solutions of equation AA_y =C(y) that can be found as parameterization of the intersection of hyperbolic a^2 + bc -1=0 and plane 2aC_1 + cC_2 + bC_3=0. Similarly we can describe the set of solutions of equation A_xA = B(x).

Surprisingly, that to get the normal two variable fields at is easier use 3 by 3 matrices.

Examples of solutions. For instance, if C_1 equal zero (a^2 + bc =1, cC_2 + bC_3=0) and $C_2C_3 > 0$ then our equation represent the hyperbola a^2 - b^2C_3/C_2 =1. Let us put a(t) = cosh(t) , b(t) =$\sqrt{C_2/C_3}\sinh(t)$.

In this case

$$A = \begin{pmatrix} \cosh(t) & \sqrt{C_2/C_3}\sinh(t) \\ -\sqrt{C_2/C_3}\sinh(t) & -\cosh(t) \end{pmatrix}$$

, where t = t(x,y). $A_t = \begin{pmatrix} \sinh(t) & \sqrt{C_2/C_3}\cosh(t) \\ -\sqrt{C_2/C_3}\cosh(t) & -\sinh(t) \end{pmatrix}$ and $A_y = A_t t_y$. Therefor

$$AA_y = t_y \begin{pmatrix} 0 & \sqrt{C_2/C_3} \\ \sqrt{C_3/C_2} & 0 \end{pmatrix}.$$

So our equation AA_y = C, where C = $\begin{pmatrix} 0 & C_2 \\ C_3 & 0 \end{pmatrix}$ can be transformed into system $t_y\sqrt{C_2/C_3} = C_2$, $t_y\sqrt{C_3/C_2} = C_3$ or it means that really we have one equation $t_y = \sqrt{C_3C_2}$.

And after integration t $=\int \sqrt{C_3C_2}dy$ + R(x), where R(x) is an arbitrary function from x or constant ($C_2 = C_2(y)$, $C_3 = C_3(y)$ are function of y or constants). In case when C_1 equal zero and $C_2C_3 < 0$ we have an ellipse and solution

$$A = \begin{pmatrix} \cos(t) & \sqrt{-C_2/C_3}\sin(t) \\ -\sqrt{-C_2/C_3}\ sin(t) & -\cos(t) \end{pmatrix}$$

, where t $=\int \sqrt{C_3C_2}$ dy + R(x).

Suppose C_1 is nonzero function or constant. In this case we have

$$\begin{pmatrix} -\cosh(t) + \sinh(t) & L^{-1}(-\sinh(t) + \cosh(t)) \\ 2L\sinh(t) & \cosh(t) - \sinh(t) \end{pmatrix}.$$

6. Final step: transformation continuous case into discrete.

We can use D- fields and (E, D) - fields defined in solid space. (E,D) - fields can be generated from D-fields by chose finite number points and then "over blowing" points in domains with values on the bounds of domain equal to value in given points. Then we must just define field equal to the E for all internals points of domains.

When product potential D-fields will be chosen we have to immerse graph into solid space and, if we wont get (E, D) - field, "over blowing" same nodes of immersed graph. Then for getting discrete values of field on the edges we have to take P1 (limit for sets of partitions of odd number points on edge) or P2 (limit for sets of partitions of even number points on edge) product integral along all edges. But for every close path on the graph the number edges where we were taken P1 integral must be even! Because of E - fields are product potential for P2- product integral. We can you different combination of P1 and P2 integral along different edges. Some times this procedure gives us many combinations of product potential system of marking of edges.

Example.

For a triangle we can get only two combinations: for all three edges were take P_2 - integral and for two edges were take P1 - integral and for one age was take P_2 - integral.

We define the function of relation f(k, j) = detG(k, j), where G(k, j) equal P1 or P2 integral for D- or (E,D)- field along edge (k, j). It is clear that to us that number negative values of relation equal the number edges with negative determinant of reactions, but this number must be positive.

So, all product potential systems automatically are balanced systems.

And for product potential system structural theorem for full graph of relation is true: for product potential system exist a maximum two antagonistic groups.

How arbitrary D field can be transformed into product potential field?

Mechanism look very easy for D - field defined on the solid domain. What does it mean to product potential (we use P_2 product integral)? It means that all values must belong to plane Aa + Bb + Cc =0 (where A=2C_1, B=C_3, C=C_2)in given domain. So we have systems a^2 + bc= 1 and Aa + Bb + Cc =0, where (a, b, c)belongs to small domain, that really is ellipse or hyperbola. Then we make gradient system on the surface a2+bc=1 that has direction to curve - ellipse or hyperbola (a2 + bc= 1and Aa + Bb + Cc =0).

Right now we do differential equations that transform initial fields to product potential field.

To make picture more understandable we change system coordinate.

Let x=a, y=(b+ c)/2, z=(b - c)/2. Equation a^2 + bc = 1 will be transformed into more familiar x^2 +y^2 - z^2=1 (hyperboloid). According main theorem (item 2.5) potential fields are fields are intersection of hyperboloid and plane. We get plane z=0 (circuit). Vector fields on the hyperboloid will define differential equation

$$dx/dt = (-t\sinh(t))/\sqrt{1+k^2}$$

$$dy/dt = (-tk\sinh(t))/\sqrt{1+k^2}$$

$$dz/dt = -t\cosh(t)$$

, where (k,t) is internal coordinate on the hyperboloid and x=$cosh(T)/\sqrt{1+k^2}$, y=kcosh(t)/ $\sqrt{1+k2)}$, and z=$sinh$(t).

The our program has been realized. We can submerse the graph without intersection into N- dimensional space and then we can find the non-constant social potential fields defined in the same N-dimensional Euclidean area.

Conclusion.

We proved that any product potential system on the graph is a balanced system in the social sense. Therefore, we can say that in our model being balanced is identical to it being a product potential system. So the structural theorem for a balanced group (stating that a balanced group has at most two antagonistic subgroups) is true for all product potential systems containing full graphs of relations.

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